Planification de mouvements pour les systèmes non-holonomes et étude de la contrôlabilité spectrale pour les équations de Schrödinger linéarisées

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献给父亲，母亲，南

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Chapitre 1

Introduction générale

La théorie du contrôle analyse les propriétés des systèmes commandés, c'est-à-dire des systèmes dynamiques sur lesquels on peut agir au moyen d'une commande. Les origines de ces systèmes sont diverses : physique, mécanique, chimie, électricité, électronique, biologie, économie, etc. Ces systèmes sont souvent modélisés par des équations différentielles, aux dérivées partielles, discrètes ou avec retard. Elles peuvent être déterministes ou stochastiques selon les phénomènes modélisés.

Un des buts en théorie du contrôle est de concevoir des commandes adéquates pour modifier le comportement de ces systèmes afin d'atteindre un objectif souhaité. Quand la commande est indépendante de l'état du système, on parle de contrôle en *boucle ouverte*. Si la commande en dépend explicitement, on parlera alors de contrôle en *boucle fermée*, ou de *feedback*. Une des questions fondamentales est la suivante : comment amener un système commandé d'un état initial donné à un certain état final, tout en respectant des contraintes éventuelles ? Pour répondre à cette question, on doit d'abord s'assurer qu'il est *possible* de passer du premier état au second pour le système en question. C'est le problème de la contrôlabilité. Si c'est possible, on essaie ensuite de construire une loi de commande qui réalise le passage souhaité. Il s'agit alors du problème de la planification de mouvements.

Dans ce mémoire de thèse, nous nous plaçons dans le cadre déterministe et nous intéressons d'une part à une classe de systèmes non-linéaires en dimension finie appelés systèmes non-holonomes, et d'autre part à des systèmes en dimension infinie modélisés par des équations aux dérivées partielles linéaires. Notre étude s'articule autour du problème de la planification de mouvements en *boucle ouverte* pour ces systèmes.

Planification de mouvements pour les systèmes non-holonomes En dimension finie, nous étudions le problème de la planification de mouvements pour les systèmes non-holonomes. Ce sont des systèmes affines en contrôle et sans dérive de la forme :

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x), \tag{1.1}$$

où l'état x appartient à $\mathbb{R}^n$, les X_i sont des champs de vecteurs, et $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ est la commande. Nous nous intéressons à la question suivante.

Question 1.1 (Planification de mouvements)**.** *Soit x^{initial} et x^{final} deux états quelconques du système* (1.1)*. Comment construire une loi de commande u qui amène ce système de x^{initial} à x^{final} ?*

Notons d'abord que, pour cette classe de systèmes, la contrôlabilité est garantie par la condition du rang que nous rappelons brièvement au chapitre 2. Un certain nombre de méthodes de planification existent déjà dans la littérature. Parmi ces méthodes, deux catégories nous semblent particulièrement importantes pour les systèmes non-holonomes. La première regroupe les méthodes basées sur l'algèbre de Lie associée au système non-holonome (cf. [76, 63, 98, 68]) ; la deuxième est basée sur la méthode de continuation (cf. [97, 25]). Nous essayons de dégager les idées principales de ces méthodes et d'expliquer pourquoi aucune d'elles

ne permet de résoudre le problème de planification de façon satisfaisante pour les systèmes non-holonomes quelconques.

Inspirés de ces méthodes, nous avons apporté une double contribution au problème de la planification de mouvements pour les systèmes non-holonomes. Dans l'esprit des méthodes basées sur l'algèbre de Lie, nous avons conçu un nouvel algorithme qui résout complètement le problème dans un cadre général. Nous avons également proposé une implémentation numérique de la méthode de continuation qui fournit des solutions satisfaisantes au problème de la planification du roulement sur le plan, un exemple classique de systèmes non-holonomes à deux entrées.

Nous présentons notre algorithme général au chapitre 3. Il s'agit d'une méthode par approximation : on approxime d'abord le système dans un sens approprié ; on construit ensuite une commande qui contrôle *exactement* le système approché ; on applique enfin cette commande au système d'origine. Si une certaine propriété de *contraction* est satisfaite, on peut espérer, en itérant ce procédé, amener le système arbitrairement proche du but, à condition que ce dernier ne soit pas trop loin du point de départ. Il faut ensuite être capable de *globaliser* ce procédé local pour en obtenir une méthode globale de planification. Notons que cette idée d'approximation et itération est déjà présente dans [63, 54], mais on y trouve de sérieuses limitations de nature différente. Notre méthode en constitue une extension complète dans le sens où elle est parfaitement constructive et explicite, et sa convergence globale est démontrée sous la seule hypothèse que la condition du rang est vérifiée.

Trois étapes sont nécessaires à la construction de l'algorithme et chacune d'elles constitue l'une de nos principales contributions. La première est une procédure de relèvement explicite et algébrique qui permet de transformer un système non-holonome avec singularités en un système non-holonome libre, donc équi-régulier, dont la résolution fournit une solution au problème de planification pour le système d'origine. A partir du système ainsi désingularisé, on peut construire un système approché qui varie continûment par rapport au point d'approximation. Cette continuité, qui ne peut pas avoir lieu si le système possède des singularités, est cruciale pour l'étape de globalisation. La deuxième contribution est une méthode exacte de planification pour le système approché à l'aide de commandes sinusoïdales. Cette méthode constitue une généralisation complète de [76], elle fournit en particulier une solution exacte et régulière pour les systèmes nilpotents. Enfin, la troisième contribution est un algorithme effectif de type *région de confiance* qui permet de globaliser la méthode locale. Notons que les méthodes classiques de région de confiance, bien connues en optimisation, ne peuvent pas s'appliquer ici car la distance naturelle dans le cadre des systèmes non-holonomes est impossible à calculer explicitement, contrairement à la distance euclidienne dans $\mathbb{R}^n$.

En ce qui concerne la deuxième catégorie de méthodes basées sur la continuation, il semble extrêmement difficile de démontrer théoriquement sa convergence globale, sauf pour des systèmes très particuliers. Cependant, elles sont très efficaces et relativement faciles à implémenter en pratique. Nous avons adopté cette approche pour résoudre le problème de la planification du roulement d'un corps strictement convexe sur un plan, qui est un exemple de systèmes non-holonomes à deux entrées. Au chapitre 4, les différentes étapes de l'implémentation sont décrites en détail et les performances de cette méthode sont illustrées à l'aide d'exemples variés.

Etude de contrôlabilité spectrale des équations de Schrödinger linéaires Dans la deuxième partie de notre travail, nous nous sommes intéressés au contrôle des équations de Schrödinger linéaires en dimension 2 et 3. La question ici est celle de la contrôlabilité, plus précisément la contrôlabilité spectrale, c'est-à-dire la possibilité d'amener le système d'un état stationnaire à un autre état stationnaire.

Rappelons d'abord l'origine physique des systèmes étudiés. Nous considérons une particule quantique dans un puits de potentiel infini de $\mathbb{R}^n$ avec $n = 2$ ou 3, soumis à un champ électrique u, uniforme dans l'espace et dépendant uniquement du temps t. La forme du puits correspond à un domaine Ω de $\mathbb{R}^n$. Ce système est modélisé par une fonction d'onde

$$\begin{array}{rccc} \psi: & \mathbb{R}_+ \times \Omega & \to & \mathbb{C} \\ & (t,q) & \mapsto & \psi(t,q), \end{array}$$

où $|\psi(t,q)|^2 dq$ represente la probabilité de trouver la particule dans un volume de taille dq autour de q à l'instant t. Sous l'hypothèse de l'approximation du moment dipolaire, la fonction d'onde ψ satisfait l'équation de Schrödinger suivante :

$$\begin{cases} i\dfrac{\partial \psi}{\partial t}(t,q) = -\Delta\psi(t,q) - \langle u(t), \mu(q)\rangle \psi(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1.2}$$

où μ est le moment dipolaire et $\langle ., . \rangle$ désigne le produit scalaire euclidien de $\mathbb{R}^n$.

Le système (1.2) est un système commandé *non-linéaire* en dimension *infinie* dont l'état est représenté par la fonction d'onde ψ et la commande est le champ electrique u. Une approche classique pour démontrer la contrôlabilité locale autour d'une trajectoire de référence d'un système non-linéaire comme (1.2) est de prouver d'abord la contrôlabilité du système linéarisé autour de la trajectoire de référence, et d'obtenir ensuite la contrôlabilité locale du système non-linéaire par un théorème d'inversion locale. Si le système linéarisé n'est pas contrôlable, on peut utiliser la méthode de retour introduite par Coron et qui consiste à linéariser le système autour d'une autre trajectoire bien choisie dont le système linéarisé correspondant est contrôlable (cf. [33, 32]). Par conséquent, afin de démontrer des résultats de contrôlabilité du système (1.2), il est naturel de commencer par le linéariser autour de trajectoires "simples", par exemple autour de celles qui correspondent à une commande identiquement nulle, et de démontrer des propriétés de contrôlabilité pour ces systèmes linéaires. On vérifie facilement que les *états stationnaires* définissent de telles trajectoires, i.e, pour $k \in \mathbb{N}^*$, chaque état stationnaire $\psi_k(t,q) := \phi_k(q) e^{-i\lambda_k t}$ est une solution de (1.2) avec $u \equiv 0$, où $(\phi_k)_{k\in\mathbb{N}^*}$ sont les fonctions propres du Laplacien de Dirichlet sur Ω et $(\lambda_k)_{k\in\mathbb{N}^*}$ sont les valeurs propres correspondantes, comptées avec multiplicité. Dans le cas où $k = 1$, ψ_1 est appelé *l'état fondamental*. En linéarisant le système (1.2) autour de l'état fondamental, nous obtenons le système (1.3)

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t}(t,q) = -\Delta\Psi(t,q) - \langle v(t), \mu(q)\rangle \psi_1(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{1.3}$$

Nous avons étudié la contrôlabilité spectrale en temps fini du système (1.3) ainsi que celle d'un système analogue. Les résultats obtenus sont présentés au chapitre 5. Nous avons démontré que la condition suivante est nécessaire : si Ω est la forme du puits de potentiel, alors pour chaque valeur propre λ du Laplacien de Dirichlet sur Ω, la projection du moment dipolaire sur les vecteurs propres correspondant à λ forment une famille libre dans $\mathbb{R}^n$. Cette condition est une généralisation du critère de Kalman, et nous la notons donc (Kal).

En dimension 3, nous avons montré que si le moment dipolaire garde une direction fixe, alors les systèmes étudiés ne sont jamais contrôlables en temps fini entre les états stationnaires.

En dimension 2, en revanche, nous avons démontré qu'il existe un temps minimal $T_{min}(\Omega) > 0$ pour obtenir la contrôlabilité spectrale, i.e.,

- si $T > T_{min}(\Omega)$ et que la condition (Kal) est vérifiée sur Ω, les systèmes étudiés sont contrôlables en temps T entre les états stationnaires ;
- si $T < T_{min}(\Omega)$ et que le moment dipolaire garde une direction fixe, les systèmes étudiés ne sont pas contrôlables en temps T entre les états stationnaires.

Notons que la condition (Kal) pour un domaine $\Omega \subset \mathbb{R}^2$ donné est difficile à vérifier par un calcul explicite. Par conséquent, nous avons donné une condition nécessaire et suffisante sur le moment dipolaire μ pour que (Kal) soit génériquement vraie sur les domaines de $\mathbb{R}^2$. Le fait que la fonction μ ne soit jamais localement constante est clairement une condition nécessaire. En effet, si on considère un domaine ouvert $\Omega \subset \mathbb{R}^2$ tel que μ soit constante sur Ω, il est évident que (Kal) n'est pas vérifiée sur Ω car les fonctions propres sont orthogonales au sens de L^2. Notre principale contribution est de prouver que cette condition est aussi suffisante. La preuve par contradiction repose sur le lemme de Baire, la différentiation de forme, et la représentation intégrale des solutions des équations du type Helmholtz. Cette stratégie peut s'appliquer pour démontrer d'autres résultats de généricité par rapport aux domaines dans le cadre des équations aux dérivées partielles linéaires.

Première partie

Planification de mouvements pour les systèmes non-holonomes

Chapitre 2

Résultats connus : contrôlabilité et méthodes de planification existantes

Sommaire

Nous nous intéressons au problème de planification de mouvements pour une classe de systèmes non-linéaires, les systèmes non-holonomes. Ces systèmes sont de la forme

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x), \qquad x \in \Omega,\ u = (u_1, \ldots, u_m) \in U. \tag{2.1}$$

Ici, Ω est un sous-ensemble ouvert et connexe de $\mathbb{R}^n$ ou une variété connexe de dimension n, U est un sous-ensemble de $\mathbb{R}^m$, les X_i sont des champs de vecteurs de classe C^∞ définis sur Ω et la loi de commande $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$ est une fonction intégrable définie sur un intervalle $[0, T]$ et à valeurs dans U. L'ensemble des lois de commandes sera noté par $\mathcal{U}$. Nous utilisons $x(\cdot, a, u)$ pour désigner la trajectoire de (2.1) égale à $a \in \Omega$ à l'instant $t = 0$ et correspondant à la loi de commande u.

Dans ce chapitre, nous présentons les résultats connus sur la planification de mouvements pour les systèmes non-holonomes : nous rappelons d'abord, en section 2.1, le théorème de Chow qui garantit la contrôlabilité de ces systèmes ; nous décrivons ensuite quelques méthodes classiques de planification pour les systèmes non-holonomes. Celles qui constituent le point de départ de nos travaux sont exposées avec plus de détails dans les sections 2.2 et 2.3, les autres méthodes, bien que très importantes, ne sont présentées que de façon succincte en section 2.4.

2.1 Contrôlabilité

Nous utilisons $L(X)$ et $L(x)$ pour désigner respectivement l'algèbre de Lie engendrée par les champs de vecteurs $X_1, \ldots, X_m$ et son évaluation en un point $x \in \Omega$ (cf. [19]).

Définition 2.1 (Condition du rang)**.** Les champs de vecteurs de $X_1 \ldots, X_m$ vérifient *la condition du rang* si, pour tout $x \in \Omega$, on a $L(x) = \mathbb{R}^n$.

Théorème 2.1 (Chow). *Si les champs de vecteurs $X_1, \dots, X_m$ vérifient la condition du rang et que U contient un voisinage de l'origine dans $\mathbb{R}^m$, alors le système (2.1) est contrôlable.*

Remarque 2.1. Le théorème 2.1 est connu sous le nom de *théorème de Chow*. Le lecteur est invité à consulter par exemple [30], [11], [55, Chapter 2], [2, Chapter 5] ou [73, Section 2.4] pour une preuve de ce théorème. Il donne une condition suffisante de contrôlabilité pour les systèmes non-holonomes. Si on suppose de plus que les champs de vecteurs $X_1, \dots, X_m$ sont *analytiques*, alors les hypothèses du théorème (2.1) sont aussi nécessaires (cf. [2, Section 5.5]).

Une conséquence du théorème 2.1 est le résultat suivant.

Corollaire 2.2. *Sous les hypothèses du théorème 2.1, pour tout ouvert connexe $M \subset \Omega$, deux points quelconques de M peuvent être joints par une trajectoire du système qui reste dans M.*

Le corollaire 2.2 nous garantit que la présence des contraintes (obstacles) dans l'espace d'états Ω ne change pas fondamentalement la propriété de contrôlabilité à condition que la connexité de Ω soit préservée. Ce résultat, dit de *contrôlabilité locale*, est particulièrement important en robotique. En effet, imaginons un robot évoluant au milieu d'obstacles. On se demande à quelle condition sur les obstacles le système reste contrôlable et, le cas échéant, comment trouver une trajectoire admissible reliant deux états donnés. On commence d'abord par représenter les obstacles comme des zones interdites de l'espace d'états, c'est-à-dire par un fermé $F \subset \Omega$. Grâce au corollaire 2.2, il suffit que $\Omega \setminus F$ soit connexe pour que deux points quelconques de $\Omega \setminus F$ soient joignables sans que le robot entre en collision avec un obstacle. Trouver une trajectoire admissible s'effectue en deux étapes :

(i) trouver une courbe reliant le point initial et le point final (ce n'est en général pas une trajectoire du système non-holonome) ;

(ii) approximer cette courbe par une trajectoire du système, c'est-à-dire trouver une trajectoire restant dans un voisinage tubulaire de la courbe et reliant le point initial et le point final.

Cette stratégie est possible : l'étape (i) est réalisable dès que $\Omega \setminus F$ est connexe. Pour l'étape (ii), comme $\Omega \setminus F$ est ouvert, il existe un voisinage connexe M de la courbe contenu dans $\Omega \setminus F$ et le corollaire 2.2 garantit l'existence d'une trajectoire joignant le point initial et le point final.

Par conséquent, nous supposons dans les chapitres 2, 3 et 4 que

(A) la condition du rang est toujours satisfaite ;

(B) les systèmes évoluent dans un environnement sans obstacles.

2.2 Méthodes basées sur les crochets de Lie

Nous présentons dans cette section les méthodes basées sur les crochets de Lie. L'idée est d'utiliser les crochets de Lie pour engendrer toutes les directions dans l'espace tangent à Ω en chaque point. Le calcul suivant en montre le principe.

Soit $p \in \mathbb{R}^n$ et X_1, X_2 deux champs de vecteurs sur $\mathbb{R}^n$. Considérons la courbe $\gamma(t)$ définie par $\gamma(t) := e^{-tX_2} \circ e^{-tX_1} \circ e^{tX_2} \circ e^{tX_1}(p)$, pour t suffisamment petit, où e^{tX_i} désigne le flot du champ X_i à l'instant t. Il est bien connu que

$$\gamma(t) = p + t^2[X_1, X_2](p) + o(t^2) \tag{2.2}$$

(cf. par exemple [75, pages 323-324] ou [16, page 30]). On crée la direction engendrée par le crochet $[X_1, X_2]$ grâce à la commutation de flots. Les trois méthodes décrites dans cette section utilisent des techniques différentes pour reproduire ce phénomène.

2.2.1 Méthode exacte de Murray-Sastry pour les systèmes chainés

Nous présentons dans cette section une méthode exacte de planification, due à R. M. Murray et S. S. Sastry [76], valide pour une classe particulière de systèmes non holonomes, appelés *systèmes chainés*. Cette méthode utilise des commandes sinusoïdales de fréquences entières vérifiant des conditions d'annulation particulières. Signalons au passage que l'idée d'utiliser cette famille de commandes trouve son origine dans l'article de Brockett [20].

Afin de ne pas alourdir les notations, nous avons choisi de présenter la méthode pour $m = 2$, c'est à dire pour les systèmes de la forme

$$\dot{x} = u_1 X_1(x) + u_2 X_2(x), \qquad \text{avec } x \in \mathbb{R}^n \text{ et } u \in \mathbb{R}^2. \tag{2.3}$$

L'idée est de contrôler d'abord une forme *canonique* pour les systèmes du type (2.3) et de trouver ensuite une transformation adéquate qui permet de mettre les systèmes généraux sous la forme canonique. Une forme particulièrement adaptée au problème de planification a été proposée par Grayson et Grossman dans [41]. Cette forme canonique jourera un rôle important dans la méthode générale de planification que nous présentons dans le chapitre 3 et qui fera l'objet d'un exposé plus détaillé au paragraphe 3.3.2. Nous nous contentons de considérer ici un exemple. Pour $m = 2$ et $n = 8$, la forme canonique de Grayson-Grossman s'écrit comme suit :

$$\begin{array}{rcll}
\dot{x}_1 &=& u_1, & X_1, \\
\dot{x}_2 &=& u_2, & X_2, \\
\dot{x}_3 &=& x_1 u_2, & X_3 := [X_1, X_2], \\
\dot{x}_4 &=& \frac{1}{2} x_1^2 u_2, & X_4 := [X_1, [X_1, X_2]], \\
\dot{x}_5 &=& x_1 x_2 u_2 & X_5 := [X_2, [X_1, X_2]] \\
\dot{x}_6 &=& \frac{1}{6} x_1^3 u_2, & X_6 := [X_1, [X_1, [X_1, X_2]]], \\
\dot{x}_7 &=& \frac{1}{2} x_1^2 x_2 u_2, & X_7 := [X_2, [X_1, [X_1, X_2]]], \\
\dot{x}_8 &=& \frac{1}{2} x_1 x_2^2 u_2, & X_8 := [X_2, [X_2, [X_1, X_2]]].
\end{array} \tag{2.4}$$

L'objectif est de contrôler le système (2.4) composante par composante. Le point crucial de cette stratégie est de s'assurer que si on déplace une composante x_i pendant une période $[0, T]$, alors aucune des composantes x_k avec $k < i$ ne soit déplacée, c'est-à-dire $x_k(0) = x_k(T)$. La propriété suivante des fonctions sinusoïdales permet d'envisager une telle stratégie : quand on intègre les fonctions sinusoïdales à fréquences entières sur $[0, 2\pi]$, on a

$$\int_0^{2\pi} \sin n\, t\, dt = 0 \quad \text{pour tout } n \in \mathbb{N}, \tag{2.5}$$

$$\int_0^{2\pi} \cos n\, t\, dt = \begin{cases} 0 & \text{si } n \in \mathbb{N} \text{ et } n \neq 0 \\ 1 & \text{si } n = 0 \end{cases}. \tag{2.6}$$

Dans [76], les auteurs ont proposé d'utiliser $u_1 := a \cos \omega_1 t$ et $u_2 := b \sin \omega_2 t$, où les paramètres a et b sont à régler en fonction des états initial et final et les fréquences entières ω_1 et ω_2 sont à choisir en fonction de la composante à contrôler. Grâce à la forme triangulaire et polynomiale du système (2.4), toutes les composantes s'écrivent comme une combinaison linéaire des sinusoïdes à fréquences entières (fonction de ω_1 et ω_2). Les relations (2.5) et (2.6) impliquent qu'il faut choisir ω_1 et ω_2 de sorte qu'un terme en cosinus avec fréquence égale à 0, c'est-à-dire un terme constant, apparaisse dans la dynamique $\dot{x}_i$ si la composante x_i est à déplacer. Par ailleurs, aucune composante x_k avec $k < i$ ne doit vérifier cette condition afin de garantir $x_k(0) = x_k(2\pi)$. Par conséquent, il est essentiel de connaître les fréquences qui interviennent dans chaque composante en fonction de ω_1 et de ω_2.

Il est facile de voir que l'on peut toujours choisir ω_1 et ω_2 pour contrôler séparément les composantes x_i avec $i = 1, \ldots, 5$ sans entraîner de déplacement sur x_k avec $k < i$. Le problème se complique quand on considère les composantes x_6, x_7 et x_8. Afin d'éviter les déplacements (au terme d'une période de contrôle $[0, 2\pi]$) sur $x_1, \ldots, x_5$, les fréquences ω_1 et ω_2 doivent vérifier les relations

$$\omega_1 \pm \omega_2 \neq 0, \quad \omega_1 \pm 2\omega_2 \neq 0 \quad \text{et} \quad 2\omega_1 \pm \omega_2 \neq 0.$$

Par un calcul direct, on sait également que les fréquences suivantes interviennent dans $x_6 - x_8$:

$$\begin{array}{lllll} x_6 : & \omega_1 \pm \omega_2, & 3\omega_1 \pm \omega_2, & & \\ x_7 : & 2\omega_1 \pm 2\omega_2, & 2\omega_1,\ 2\omega_2, & \omega_1, & \\ x_8 : & \omega_1 \pm 3\omega_2, & \omega_1 \pm \omega_2, & \omega_2. & \end{array}$$

En conséquence, si on choisit $\omega_2 = 3\omega_1$ (resp. $\omega_1 = 3\omega_2$), on peut contrôler x_6 (resp. x_8) sans entraîner de déplacements sur $x_1 - x_5$. Cependant, il est impossible de réaliser une telle opération pour x_7. En effet, on est tenté d'imposer $2\omega_1 = 2\omega_2$ pour contrôler x_7, mais ceci implique $\omega_1 = \omega_2$, entraînant *inévitablement*[1] un déplacement de la composante x_3[2].

Face à cette difficulté plutôt *intrinsèque*[3], les auteurs ont dit dans [76] : *It may still be possible to steer the system using combinations of sinusoids at different frequencies for each input or using more complicated periodic functions [...] Rather than explore the use of more complicated inputs for steering nonholonomic systems, we consider instead a simpler class of systems.*

Considérons un système de la forme suivante, appelé *système chainé*, ou plus précisément *1-chainé* :

$$\begin{array}{rcl} \dot{x}_1 &=& u_1 \\ \dot{x}_2 &=& u_2 \\ \dot{x}_3 &=& x_2 u_1 \\ \dot{x}_4 &=& x_3 u_1 \\ &\vdots& \\ \dot{x}_n &=& x_{n-1} u_1. \end{array} \tag{2.7}$$

Par un calcul direct, on peut montrer que l'algorithme suivant permet de contrôler tous les systèmes de la forme (2.7). Le lecteur pourra trouver une preuve détaillée dans [75, pages 363 – 366].

Remarque 2.2. Expliquons un peu le rôle déterminant que joue la structure chainée dans cet algorithme. En effet, si on choisit

$$u_1 := a \sin \omega_1 t, \qquad u_2 := b \cos \omega_2 t,$$

à cause de la structure particulière du système (2.7), quelle que soit la composante x_i avec $i \geq 3$, sa dynamique $\dot{x}_i$ s'écrit comme une combinaison linéaire de sinusoïdes dont les fréquences sont de la forme

$$\omega_2 + \gamma\, \omega_1, \qquad \text{où} \ \ \gamma \in \mathbb{Z}, \quad |\gamma| \leq i - 2. \tag{2.8}$$

1. On vérifie sans difficulté qu'il y a effectivement un terme en cosinus avec fréquence 0, c'est à dire un terme constant, qui apparaît dans $\dot{x}_3$.

2. Un calcul direct montre que modifer la phase de u_1 et u_2, c'est à dire démarrer avec $u_1 = a \cos \omega_1 t$ et $u_2 = b \sin \omega_2 t$ par exemple, ne résout pas le problème.

3. *Intrinsèque* dans le sens où ce phénomène dit de *résonance* n'est pas spécifique au système (2.4). On verra au chapitre suivant qu'il est impossible de séparer les composantes d'un système non-holonome quelconque en n'utilisant que deux fréquences.

Algorithm 1 Contrôler un système chainé

1: Déplacer x_1 et x_2 à leurs positions souhaitées.

2: Pour toute composante x_{k+2} avec $k \geq 1$, déplacer x_{k+2} en utilisant

$$u_1 := a \sin t \quad \text{et} \quad u_2 := b \cos kt$$

où les paramètres a et b vérifient

$$x_{k+2}(2\pi) - x_{k+2}(0) = \frac{(a/2)^k b}{k!} \cdot 2\pi.$$

Si on note γ_i la valeur maximale de γ pour la composante x_i, il est facile de voir que $\gamma_i = i-2$. La suite (γ_i) est donc *monotone*. Par conséquent, quand on veut déplacer la composante x_{k+2} avec $\omega_1 = 1$ et $\omega_2 = k$, la relation (2.8) nous empêche d'avoir une fréquence nulle dans la dynamique des composantes x_i avec $i < k+2$. Cette propriété n'est évidemment pas garantie pour les systèmes de la forme (2.4).

Remarque 2.3. Nous donnons en section 3.5 une construction générale qui permet de contrôler les systèmes sous forme canonique au sens de Grayson et Grossman [41] par des commandes sinusoïdales, ce qui généralise complètement la méthode proposée dans [76].

Le lecteur est invité à consulter [76, Section 4.2] pour un traitement détaillé sur les systèmes chainés (2-chaines et multi-chaines). Dans [76, Section 4.4], les auteurs ont donné une condition suffisante (Proposition 11) permettant de déterminer si un système est équivalent à une $1-$chaine par bouclage statique. La preuve de la proposition 11 dans [76] est constructive.

2.2.2 Méthode par approximation nilpotente de Lafferriere-Sussmann

Nous présentons dans ce paragraphe une méthode générale de planification développée par G. Lafferriere et H. J. Sussmann dans [63]. Cette méthode est exacte pour les systèmes nilpotents. Elle fournit une solution approchée [4] dans le cas général.

Cas nilpotent Rappelons qu'une algèbre de Lie engendrée par des champs de vecteurs est dite *nilpotente d'ordre k* si tous les crochets de Lie de longueur supérieure ou égale à $k+1$ sont nuls et que ceux de longueurs k ne sont pas tous nuls.

Nous supposons dans ce paragraphe que $L(X)$ est nilpotente d'ordre k et que $\{X_1, \ldots, X_r\}$ sont les éléments de la base Hall engendrée par $\{X_1, \ldots, X_m\}$ [5]. Dans ce cas, on peut montrer qu'il existe r fonctions $h_1^u, \ldots, h_r^u$ définies sur $[0, T]$ telles que $S^u(t)$ le flot du système (2.1) associé à la commande u s'écrit sous la forme

$$S^u(t) = e^{h_r^u(t)X_r} \circ \cdots \circ e^{h_1^u(t)X_1}, \tag{2.9}$$

où $e^{h_i X_i}$ désigne le flot du champ X_i à l'instant h_i. Le r-uplet $(h_1(\cdot), \ldots, h_r(\cdot))$ est appelé *coordonnées de Hall* ou *coordonnées exponentielles de deuxième espèce.*

4. Solution approchée : si nous souhaitons amené le système (2.1) de p à q, cette méthode peut nous fournir une commande qui amène le système arbitrairement proche du but q.

5. Afin de dégager les idées principales de la méthode sans introduire trop de formalisme, nous avons omis de donner une définition formelle de la base de Hall. Un exposé rigoureux à ce sujet sera donné au paragraphe 3.3.1.

En définissant l'application *entrée-sortie (end-point map)* partant du point p en coordonnées exponentielles

$$\Phi_p : (u_1, \ldots, u_m) \mapsto (h_1(T), \ldots, h_r(T)), \tag{2.10}$$

nous pouvons considérer la méthode de planification proposée par Lafferriere et Sussmann comme une procédure effective qui exprime d'abord le point but q en coordonnées exponentielles, puis qui inverse l'application Φ_p.

Algorithm 2

1: Exprimer le but q en coordonnées exponentielles, i.e., trouver le r-uplet $(q_1, \ldots, q_r)$ tel que $q = e^{q_r X_r} \circ \cdots \circ e^{q_1 X_1}(p)$.
2: Calculer une commande u telle $\Phi_p(u) = (q_1, \ldots, q_r)$.

Pour réaliser l'étape 1, on introduit *le système de contrôle étendu* :

$$\dot{x} = v_1 X_1(x) + \cdots + v_m X_m(x) + v_{m+1} X_{m+1}(x) + \cdots + v_r X_r(x), \tag{2.11}$$

où $v = (v_1, \ldots, v_r)$ est la commande.

Il est facile de trouve une commande v qui amène le système (2.11) de p à q. En effet, pour trouver une telle commande, on définit un chemin

$$\gamma : [0, T] \to \mathbb{R}^n,$$

de classe C^1 qui relie p à q. On peut par exemple choisir le segment $[p, q]$. On exprime ensuite, pour tout $t \in [0, T]$, le vecteur tangent $\dot{\gamma}(t)$ comme une combinaison linéaire de $\{X_1(\gamma(t)), \ldots, X_r(\gamma(t))\}$. Les coefficients de cette combinaison sont exactement $v_i(t)$, pour $i = 1, \ldots, r$ et $t \in [0, T]$.

Par ailleurs, notons que les flots du système étendu sont aussi de la forme (2.9). Pour calculer les fonctions $h_i(\cdot)$ correspondantes, il suffit de remplacer l'expression (2.9) dans l'équation différentielle vérifiée par le flot :

$$\begin{aligned} \dot{S}(t) &= \sum_{i=1}^{r} v_i(t) X_i(S(t)), \\ S(0) &= \mathrm{Id}. \end{aligned} \tag{2.12}$$

où Id désigne l'application *identité*.

Par un calcul un peu laborieux, on peut montrer que $h_1, \ldots, h_r$ vérifient un système triangulaire de la forme suivante :

$$\begin{aligned} \dot{h}_1 &= v_1 \\ &\vdots \\ \dot{h}_k &= Q_k(h_1, \ldots, h_{k-1}, v_1, \ldots, v_k) \\ &\vdots \\ \dot{h}_r &= Q_r(h_1, \ldots, h_{r-1}, v_1, \ldots, v_{r-1}), \end{aligned} \tag{2.13}$$

avec $h_1(0) = \cdots = h_r(0) = 0$.

Remarque 2.4. Le lecteur est invité à consulter [63, Example 1] et [75, pages 377-380] pour des exemples de calculs. Ces calculs sont basés sur la fameuse "pull-back formula" dont une preuve détaillée peut être trouvée dans [46, Appendix I].

En *intégrant* le système (2.13) sur $[0,T]$, on a (par construction) :

$$q = e^{h_r(T)X_r} \circ \cdots \circ e^{h_1(T)X_1}(p). \tag{2.14}$$

En d'autres termes, on peut poser $q_i := h_i(T)$, pour $i = 1, \ldots, r$. L'étape 1 est ainsi réalisée.

Remarque 2.5. Notons qu'en pratique l'intégration de l'équation (2.13) se fait par des méthodes de quadrature. Par conséquent, le changement de coordonnées apparu dans l'étape 1 est nécessairement *numérique*.

La deuxième étape consiste à trouver une commande u telle que le flot S^u vérifie

$$S^u(T) = e^{q_r X_r} \circ \cdots \circ e^{q_1 X_1}. \tag{2.15}$$

Le point clé est de générer séparément et successivement de droite à gauche chaque facteur (i.e. le flot engendé par chaque élément de la base de Hall) dans (2.15) en utilisant la vraie commande u. Ceci n'est pas difficile à réaliser à condition de tolérer des modifications d'ordre supérieur. Nous illustrons cette idée par un exemple.

Soit $q_3 > 0$. Supposons que l'on veuille engendrer le factor $e^{q_3 X_3}$ avec $X_3 := [X_1, X_2]$ pour un système de la form (2.3). En utilisant la commande constante par morceaux suivant :

$$(u_1, u_2) := \begin{cases} (1,0) & t \in [0,\ \sqrt{q_3}], \\ (0,1) & t \in [\sqrt{q_3},\ 2\sqrt{q_3}], \\ (-1,0) & t \in [2\sqrt{q_3},\ 3\sqrt{q_3}], \\ (0,-1) & t \in [3\sqrt{q_3},\ 4\sqrt{q_3}], \end{cases}$$

nous générons le flot $e^{-\sqrt{q_3}X_2} \circ e^{-\sqrt{q_3}X_1} \circ e^{\sqrt{q_3}X_2} \circ e^{-\sqrt{q_3}X_1}$. L'équation (2.2) implique

$$e^{-\sqrt{q_3}X_2} \circ e^{-\sqrt{q_3}X_1} \circ e^{\sqrt{q_3}X_2} \circ e^{-\sqrt{q_3}X_1} = e^{q_3 X_3} \circ e^{R}, \tag{2.16}$$

où e^R désigne la composition des flots engendrés par des crochets d'ordre supérieur. Cette idée de commutation à l'aide des commandes constantes par morceaux peut être appliquée pour engendrer les flots associés aux crochets d'ordre supérieur. Les calculs explicites reposent sur la formule de Campell-Baker-Hausdorff. Dans le cas de deux champs de vecteurs, nous avons le résultat suivant :

Proposition 2.3 (Formule de Campell-Baker-Hausdorff)**.** *Soit X_1 et X_2 deux champs de vecteurs C^∞. La composition de leurs flots est donnée par*

$$e^{X_2} \circ e^{X_1} = e^{X_1 + X_2 + \frac{1}{2}[X_1,X_2] + \frac{1}{12}([X_1,[X_1,X_2]] - [X_2,[X_1,X_2]]) \cdots}. \tag{2.17}$$

Notons ici que dans le cas nilpotent, le membre de droite de la formule (2.17) ne contient qu'un nombre fini de termes.

Théorème 2.4. *Dans le cas nilpotent, la commande u construite par la proédure présentée dans ce paragraphe amène le système du point p au point q, pour tout $(p,\ q) \in \Omega \times \Omega$.*

Cas général Si on veut généraliser la méthode décrite précédemment au cas où l'algèbre de Lie engendrée par les champs de vecteurs n'est plus nilpotente, la principale difficulté est que le nombre d'éléments dans la base de Hall est *infini*, ce qui implique que :

(i) l'équation (2.9) contient un nombre infini de termes ;

(ii) la formule (2.17) contient un nombre infini de termes.

Par conséquent, un procédé d'approximation s'impose et nous devons nous contenter d'une planification approchée dans le cas général : soit $(p,\ q) \in \Omega \times \Omega$, pour tout $e > 0$, trouver une commande u^e définie sur $[0, T]$ telle que $\|x(T,\ p,\ u^e) - q\| < e$.

Nous commençons par préciser la notion d'approximation introduite dans [63]. L'idée consiste à écrire les flots toujours en coodonnées exponentielles, puis à ne conserver que les facteurs engendrés par les crochets d'ordre inférieur ou égal à k. De façon plus concise, on dira que l'on *tronque* les flots de (2.1) à l'ordre k. Le principal avantage de cette notion d'approximation est que l'on peut parfaitement dérouler l'algorithme 2 en faisant tous les calculs, modulo les flots d'ordre supérieur à k[6]. La commande u ainsi construite n'amène évidemment pas le système exactement au point q, mais à un point $\widetilde{q}$ proche de q. Nous posons $E(p, q) := \|\widetilde{q} - q\|$. Le théorème suivant donne une estimation sur $E(p, q)$ en fonction de k et de la distance entre p et q.

Théorème 2.5. *Soit K un sous-ensemble borné de Ω. Il existe une fonction $C : [0, \infty[\to [0, \infty[$ bornée au voisinage de 0 telle que, pour tout $(p, q) \in K^2$, on a :*

$$E(p, q) \leq C(\|p - q\|)\|p - q\|^{1+\frac{1}{k}}. \tag{2.18}$$

Le résultat suivant donne une estimation uniforme sur $E(p, q)$.

Corollaire 2.6. *Si K est borné, alors il existe une constante $\Delta > 0$ telle que, pour tout $(p, q) \in K \times K$ et $\|p - q\| < \Delta$, on a*

$$E(p, q) \leq \frac{1}{2}\|p - q\|. \tag{2.19}$$

Par le théorème du point fixe, l'estimation (2.19) garantit la convergence locale (pour $\|p - q\| < \Delta$) de l'algorithme suivant.

Algorithm 3 Méthode locale de planification approchée

Require: $(p,\ q) \in K \times K, \quad e > 0$

1: $k = 0$;
2: $q^k = p$;
3: **while** $\|q^k - q\| > e$ **do**
4: Calculer u^k à l'aide de l'Algorithme 2 ;
5: $q^{k+1} := x(T, q^k, u^k)$;
6: $k = k + 1$.

Il n'est pas difficile de déduire de l'algorithme 3 une méthode globale de planification approchée. L'idée est la suivante : nous considérons un chemin $\gamma \subset K$ reliant p à q ; nous choisissons une suite de sous-buts

$$\{q_0^d = p, q_1^d, \ldots, q_n^d = q\}$$

sur γ tels que $\|q_{i-1}^d - q_i^d\| < \dfrac{\Delta}{2}$, pour $i = 0, \ldots, n$. Comme l'algorithme 3 permet d'amener le système d'un sous-but à un autre[7], en itérant cette procédure, on amène le système (2.1) arbitrairement proche de l'état final q.

6. Par abus de langage, les flots d'ordre i sont les flots engendrés par les crochets de longueur i.

7. Plus précisément, l'algorithme 3 amène le système d'un point proche d'un sous-but à un autre point proche du sous-but suivant.

Remarque 2.6. La méthode de Lafferriere-Sussmann est le premier algorithme de planification pour les systèmes de la forme (2.1) sans aucune hypothèse de structure autre que la condition du rang. Les points importants dans cette méthode sont le contrôle exact des systèmes nilpotents et la notion d'approximation nilpotente qui garantit que l'estimation (2.19) est uniformément vérifiée sur tout sous-ensemble compact de Ω.

Du point de vue pratique, les points faibles de cette méthode sont les suivants :

- le changement de coordonnées exponentielles est effectué de façon numérique (Remarque 2.5), ce qui n'est pas adapté à une méthode itérative comme l'algorithme 3 ;
- contrôler les systèmes nilpotents à l'aide de commandes constantes par morceaux et de la formule (2.17) crée des trajectoires non lisses. Afin d'obtenir une paramétrisation régulière de trajectoires, le système de contrôle est obligé de s'arrêter souvent, ce qui n'est pas très adapté aux applications en robotique ;
- la distance *critique*, Δ qui permet de construire une suite des sous-buts, est donnée dans les coordonnées exponentielles attachées au point courant à chaque itération, ce qui rend très difficile la globalisation *effective* de l'Algorithme 3.

Du point de vue théorique, les auteurs se sont placés au niveau du groupe des difféomorphismes engendrés par les flots du système pour introduire une notion d'approximation. Cette approche permet de garantir l'uniformité de leur approximation sans trop de difficulté, il suffit de tronquer les flots toujours au même ordre. Cependant, le sens de cette approximation n'est pas très clair.

2.2.3 Méthode oscillante de Liu-Sussmann

Nous présentons dans cette section la méthode introduite par H. J. Sussmann et W.-S. Liu en 1991 (cf. [98]). L'article [68] fournit tous les détails techniques sur cette méthode. Un exposé, un peu plus facile à lire, expliquant l'application de la méthode à l'aide d'un exemple concret se trouve dans [99].

L'objectif est de comprendre le lien entre les trajectoires du système

$$\dot{x} = \sum_{i=1}^{m} u_i(t) X_i(x), \tag{2.20}$$

où X_i sont des champs de vecteurs de classe C^∞ définis sur un ouvert Ω de $\mathbb{R}^n$, et celles du système étendu défini par

$$\dot{x} = \sum_{i=1}^{r} v_i(t) X_i(x), \tag{2.21}$$

où les m premiers champs sont ceux du système (2.20) et les champs suivants sont des crochets de Lie engendrés par $X_1, \ldots, X_m$. Nous supposons que pour tout $x \in \Omega$, la famille de vecteurs $\{X_1(x), \ldots, X_r(x)\}$ engendre $\mathbb{R}^n$ tout entier. Pour des raisons techniques, nous supposons également que $X_1, \ldots, X_r$ sont des éléments de la base de Hall engendrée par les champs $X_1, \ldots, X_m$.

Comme nous l'avons expliqué en section 2.2.2, il est clair que toutes les courbes lisses de Ω sont trajectoires du système étendu (2.21) et qu'il est, par conséquent, facile de résoudre le problème de planification de mouvements pour ce système. Soit $\gamma : [0, T] \to \Omega$ une trajectoire du système étendu. L'auteur a montré dans [68] qu'on pouvait construire explicitement une suite de commandes $u^j := (u_1^j, \ldots, u_m^j)$ du système (2.20) telles que les trajectoires correspondantes convergent *uniformément* en temps vers γ quand j tend vers l'infini. Ce résult permet de résoudre a fortiori le problème 1.1 de façon approchée. En effet, pour trouver une

commande qui amène le système (2.20) de x_0 à x_1, il suffit de construire la suite u^j convergeant vers une courbe γ qui relie x_0 à x_1. En choisissant j assez *grand*, la commande u^j amène (2.20) arbitrairement proche du point final x_1 (cf. [95, Algorithm 4]).

Le grand avantage de la construction proposée dans [68] est son caractère *universel* : elle est indépendante des valeurs particulières que prennent les champs de vecteurs, et ne dépend que de la structure des crochets de Lie. Par conséquent, le cadre naturel pour présenter cette construction est celui de l'algèbre de Lie libre engendrée par m indéterminées $\mathcal{X}_1, \ldots, \mathcal{X}_m$, qui fait abstraction des champs de vecteurs particuliers. Cependant, une telle abstraction rend [68] difficile à lire. Nous avons donc choisi d'expliquer cette méthode sur un exemple simple emprunté de [95] et nous nous contentons de faire quelques commentaires sur le cas général à la fin de la section.

Un exemple Considérons le système suivant

$$\dot{x} = u_1(t)X_1(x) + u_2(t)X_2(x), \qquad x \in \mathbb{R}^5, \tag{2.22}$$

où X_1 et X_2 sont deux champs de vecteurs de classe C^∞ définis sur $\mathbb{R}^5$ et la commande $u := (u_1, u_2)$ est à valeur dans $\mathbb{R}^2$. Soit $(x_0, x_1) \in \mathbb{R}^5 \times \mathbb{R}^5$. Fixons une courbe $t \to \gamma(t) \in \mathbb{R}^5$ de classe C^1 et définie sur $[0,1]$ avec $\gamma(0) := x_0$ et $\gamma(1) := x_1$. Posons $X_3 := [X_1, X_2]$, $X_4 := [X_1, [X_1, X_2]]$ et $X_5 := [X_2, [X_1, X_2]]$. Nous supposons que pour tout $x \in \mathbb{R}^5$, la famille $\{X_1(x), X_2(x), X_3(x), X_4(x), X_5(x)\}$ forme une base de $\mathbb{R}^5$. Sous cette hypothèse, il existe des fonctions lisses $v_1, \ldots, v_5$ définies sur $[0,1]$ à valeurs réelles telles que $t \to \gamma(t)$ soit la solution du problème de Cauchy

$$\dot{x} = v_1(t)X_1(x) + \cdots + v_5(t)X_5(x), \tag{2.23}$$

avec $x(0) = x_0$ (cf. Section 2.2.2). Nous cherchons à construire une suite de commande $(u^j)_{j\in\mathbb{N}}$ telle que la suite des trajectoires de (2.22) associées à $(u^j)_{j\in\mathbb{N}}$ converge uniformément en temps vers γ quand j tend vers l'infini.

L'idée est d'utiliser des commandes oscillantes qui sont des combinaisons linéaires de fonctions sinusoïdales de fréquences judicieusement choisies. Prenons trois groupes de fréquences à valeur réelle $\Omega_k = \{\omega_{k,1}, \omega_{k,2}\}$, pour $k = 1, 2, 3$, et supposons que les $\omega_{k,l}$ vérifient les conditions suivantes :

(i) (*conditions de résonance*) ;$\omega_{1,1} + \omega_{1,2} = 0$, $2\omega_{2,1} + \omega_{2,2} = 0$, $\omega_{3,1} + 2\omega_{3,2} = 0$;

(ii) (*conditions d'indépendance*) pour tout entiers α_k compris entre -2 et 2, on a

$$\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = 0,\ \omega_k \in \Omega_k,\ k = 1, \ldots, 3 \implies \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Pour toutes fonctions $\eta_1, \eta_2, \eta_{k,l}$ de classe C^1 avec $k = 1, 2, 3$ et $l = 1, 2$, nous définissons la suite de commandes suivante :

$$u_1^j(t) := \eta_1(t) + u_{1,1}^j(t) + u_{2,1}^j(t) + u_{3,1}^j(t), \tag{2.24}$$
$$u_2^j(t) := \eta_2(t) + u_{1,2}^j(t) + u_{2,2}^j(t) + u_{3,2}^j(t), \tag{2.25}$$

où

$$u_{1,1}^j(t) := j^{\frac{1}{2}}\eta_{1,1}(t)\sin\omega_{1,1}jt, \qquad u_{1,2}^j := j^{\frac{1}{2}}\eta_{1,2}(t)\cos\omega_{1,2}jt, \tag{2.26}$$
$$u_{2,1}^j(t) := j^{\frac{2}{3}}\eta_{2,1}(t)\cos\omega_{2,1}jt, \qquad u_{2,2}^j := j^{\frac{2}{3}}\eta_{2,2}(t)\cos\omega_{2,2}jt, \tag{2.27}$$
$$u_{3,1}^j(t) := j^{\frac{2}{3}}\eta_{3,1}(t)\cos\omega_{3,1}jt, \qquad u_{3,2}^j := j^{\frac{2}{3}}\eta_{3,2}(t)\cos\omega_{3,2}jt. \tag{2.28}$$

Théorème 2.7. *Avec les notations ci-dessus, la suite des trajectoires du système (2.22) associées à la suite des commandes $u^j := (u_1^j, u_2^j)$ converge uniformément vers la trajectoire du système étendu défini par*

$$\begin{aligned}\dot{x} &= \eta_1(t)X_1 + \eta_2 X_2(x) - \frac{\eta_{1,1}(t)\eta_{1,2}(t)}{2\omega_{1,1}}X_3(x) \\ &\quad - \frac{\eta_{2,1}^2(t)\eta_{2,2}(t)}{8\omega_{2,1}^2}X_4(x) - \frac{\eta_{3,1}(t)\eta_{3,2}^2}{4\omega_{3,1}\omega_{3,2}}X_5(x). \end{aligned} \tag{2.29}$$

Par conséquent, en choisissant les fonctions $\eta_1, \eta_2, \eta_{k,l}$ telles que

$$\begin{aligned}&\eta_1(t) = v_1,\ \eta_2(t) = v_2(t),\ -\frac{\eta_{1,1}(t)\eta_{1,2}(t)}{2\omega_{1,1}} = v_3(t), \\ &-\frac{\eta_{2,1}^2(t)\eta_{2,2}(t)}{8\omega_{2,1}^2} = v_4(t),\ -\frac{\eta_{3,1}(t)\eta_{3,2}^2}{4\omega_{3,1}\omega_{3,2}} = v_5(t),\end{aligned}$$

la suite des commandes $(u^j)_{j\in\mathbb{N}}$ produit une suite de trajectoires de (2.22) qui converge uniformément vers γ, ce qui termine la construction.

Il est intéressant de noter que dans le système limite (2.29) les coefficients devant X_3, X_4 et X_5 ne dépendent respectivement que de $(u_{1,1}^j, u_{1,2}^j)$, $(u_{2,1}^j, u_{2,2}^j)$ et $(u_{3,1}^j, u_{3,2}^j)$. En d'autres termes, on est capable d'engendrer séparément les directions données par les crochets de Lie d'ordre supérieur. Cette propriété peut être considérée comme une sorte de *principe de superposition non linéaire*.

Remarque 2.7. Le fait qu'une commande de type (2.26) engendre des mouvements dans la direction de X_3 est une conséquence de la condition de résonance (i). Le fait que des commandes de types (2.27) et (2.28) n'entrainent pas de déplacements dans la direction X_3 se déduit de la condition d'indépendance (ii). Rappelons que l'introduction des conditions (i) et (ii) est uniquement basée sur les équations (2.5) et (2.6).

Remarque 2.8. Une commande de type (2.26) pour j à valeur finie engendre clairement des perturbations dans les directions X_4 et X_5. Cependant, on peut montrer que ces perturbations tendent uniformément vers 0 quand j tend vers l'infini.

Pour illustrer les remarques 2.7 et 2.8, il est intéressant d'étudier les exemples donnés dans [96, Section 5] et [68, Example 3.1].

Remarque 2.9. Pour le cas général (cf. [68]), les conditions de résonance (i) et d'indépendance (ii) s'étendent sans difficulté et donnent la notion de *minimally canceling*. Afin de prouver une convergence uniforme vers 0 des termes résiduels apparus dans les crochets [8] d'ordre supérieur et causés par les commandes destinées aux crochets d'ordre inférieur (cf. Remarque 2.8), l'auteur a introduit la notion de *generalized difference*. L'estimation de ces termes résiduels en utilisant par récurrence les différences généralisées constitue le premier point technique de l'article [68].

Remarque 2.10. Une difficulté supplémentaire dans le cas général que nous n'avons pas évoquée jusqu'à présent provient du phénomène suivant. Considérons par exemple les crochets $[X_2, [X_1, [X_1, [X_1, X_2]]]]$ et $[[X_1, X_2], [X_1, [X_1, X_2]]]$, qui contiennent tous les deux 3 fois X_1 et 2 fois X_2. On peut montrer qu'il n'existe pas de fréquences résonantes permettant d'engendrer

8. Par abus de langage, ceci signifie les directions engendrées par ces crochets.

un déplacement dans l'une direction sans entrainer une perturbation sur l'autre. Autrement dit, dans le système limite correspondant, les coefficients devant ces deux directions sont liés, ce qui nous oblige de trouver un moyen d'engendrer simultanément ces deux directions (cf. [99]). La difficulté technique que ce phénomène a introduite revient à montrer l'inversibilité de certaines matrices. Sa preuve, qui repose sur une technique de relèvement, constitue le deuxième point technique de l'article [68]. Nous avons emprunté l'idée de cette preuve pour démontrer le lemma 3.25 du chapitre 3.

Remarque 2.11. Notons finalement que cette approche est difficilement utilisable en pratique car on doit prendre des commandes sinusoïdales dont les fréquences tendent vers l'infini pour que l'état final obtenu converge vers l'état final assigné. De plus, cette méthode n'est jamais exacte quel que soit le type de systèmes considérés. En particulier, un passage à la limite est nécessaire même dans le cas nilpotent.

2.3 Méthode de continuation

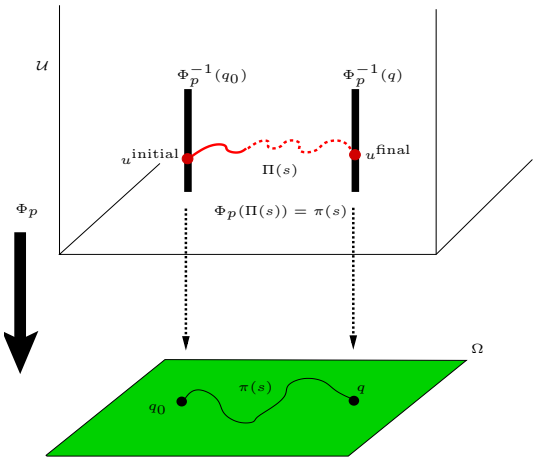
Soit $(p, q) \in \Omega \times \Omega$. Rappelons que le problème de planification consiste à trouver une commande u définie sur $[0, T]$ qui amène le système de p à q. Nous supposons dans cette section que $\mathcal{U} := L^2([0, T], \mathbb{R}^n)$. Nous considérons l'application *entrée-sortie* en temps T initialisé à p définie par

$$\Phi_p(v) := x(T, p, v), \qquad \text{pour tout } v \in \mathcal{U}.$$

Nous pouvons reformuler le problème de planification à l'aide de l'application Φ_p de façon suivante : pour tout $(p, q) \in \Omega \times \Omega$, trouver une commande $u_{p,q} \in \mathcal{U}$ telle que

$$\Phi_p(u_{p,q}) = q. \tag{2.30}$$

En d'autres termes, nous souhaitons *inverser* l'application Φ_p, ou plus précisément, nous cherchons une *inverse à droite* de Φ_p car cette application est surjective (le système est contrôlable) mais elle n'est pas injective (il existe plusieurs commandes u telle que $\Phi_p(u) = q$.). La méthode de continuation présentée dans [97, 25] fournit une procédure générale (qui n'est pas spécifique aux systèmes non-holonomes) permettant de réaliser cette inversion. L'idée est décrite dans le schéma ci-dessous.



Nous prenons une commande $u^{\text{initial}} \in \mathcal{U}$ quelconque. L'idée consiste à construire un chemin dans $\mathcal{U}$ qui permet de passer de u^{initial} à une commande u^{final} telle que $\Phi_p(u^{\text{final}}) = q$.

Posons $q_0 := \Phi_p(u^{\text{initial}})$ et choisissons un chemin $\pi : [0,1] \to \Omega$ tel que $\pi(0) := q_0$ et $\pi(1) := q$. Nous cherchons un chemin $\Pi : [0,1] \to \mathcal{U}$ tel que, pour presque tout $s \in [0,1]$,

$$\Phi_p(\Pi(s)) = \pi(s). \tag{2.31}$$

En dérivant l'équation (2.31) par rapport à s, nous obtenons

$$D\Phi_p(\Pi(s)) \cdot \frac{d\Pi}{ds}(s) = \frac{d\pi}{ds}(s), \tag{2.32}$$

où $D\Phi_p(v)$ désigne la differentielle de Φ_p au point v.

Si $D\Phi_p(\Pi(s))$ est de rang *plein*, alors toutes les solutions de l'équation différentielle

$$\frac{d\Pi}{ds}(s) = P\big(\Pi(s)\big) \cdot \frac{d\pi}{ds}(s), \tag{2.33}$$

où $P(v)$ est une inverse à droite de $D\phi_p(v)$ [9], vérifient l'équation (2.32). En conséquence, la solution du système

$$\left\{ \begin{array}{rcl} \dfrac{d\Pi}{ds}(s) & = & P\big(\Pi(s)\big) \cdot \dfrac{d\pi}{ds}(s), \\ \Pi(0) & = & u^{\text{initial}} \end{array} \right. \tag{2.34}$$

est, par construction, une solution de l'équation (2.32), ceci à condition bien sûr qu'elle existe sur *tout* l'intervalle $[0,1]$. Nous sommes donc amenés à étudier l'équation de Wazewski (2.33), appelée également l'équation de relèvement, comme une équation différentielle ordinaire définie sur $\mathcal{U}$.

Rappelons que, par construction, la commande u^{final} donnée par

$$u^{\text{final}} := \Pi(1)$$

amène le système de p à q. Pour obtenir une valeur approchée de $\Pi(1)$, il suffit d'intégrer numériquement l'équation (2.34) sur $[0,1]$ à l'aide par exemple d'un schéma d'Euler.

Par conséquent, si nous voulons appliquer la méthode de continuation pour résoudre un problème de planification, nous devons garantir les deux conditions suivantes :

(a) non dégénérescence : le chemin π doit être choisi de sorte que $D\phi_p\big(\Pi(s)\big)$ soit de rang plein pour tout $s \in [0,1]$;

(b) non explosion : l'équation de relèvement (2.34) doit posséder une solution globalement définie sur tout l'intervalle $[0,1]$, et ceci quelle que soit la condition initiale u^{initial}.

Remarque 2.12. Le point (a) garantit l'existence de $P(\Pi(s))$ pour tout $s \in [0,1]$, donc celle de l'équation (2.33). Le point (b) est crucial car nous avons besoin d'évaluer la valeur de $\Pi(\cdot)$ au point $s = 1$.

Remarque 2.13. Notons que l'existence locale ainsi que l'unicité de la solution de l'équation (2.34) est assurée dès que l'application entrée-sortie Φ_p est de classe C^2.

Il est raisonnabe de s'attendre à des difficultés pour les points singuliers de Φ_p, *i.e.*, toutes les commandes $v \in \mathcal{U}$ telles que le rang de $D\Phi_p(v)$ est strictement inférieur à la dimension de Ω. Nous notons respectivement S_p et $\Phi_p(S_p)$ l'ensemble des points singuliers de Φ_p et l'ensemble des valeurs singulières. L'application de la méthode de continuation se décompose naturellement en deux étapes :

9. Nous pouvons par exemple choisir $P(v)$ comme la pseudo-inverse de Moore-Penrose de $D\phi_p(v)$.

(i) caractériser les ensembles S_p et $\Phi_p(S_p)$;
(ii) relever à l'aide de l'équation (4.12) un chemin $\pi : [0,1] \to \Omega \setminus \Phi_p(S_p)$ à un chemin $\Pi : [0,1] \to \mathcal{U}$ qui soit globalement défini sur $[0,1]$.

Remarque 2.14. Le point (ii) qui consiste à prouver l'existence globale d'une équation différentielle fortement non linéaire est particulièrement difficile dans le cas général. Des résultats existent sous des hypothèses plus restrictives. Par exemple, dans [25], l'auteur a résolu le point (ii) en supposant que le système (2.1) est fortement Lie généré, i.e., en tout point $p \in \Omega$, $T_p\Omega$ est engendré par la valeur des champs et leurs crochets de longueur 2 évalués au point p ; dans [23], les auteurs ont prouvé que la méthode de continuation résolvait globalement le problème de planification pour les systèmes constitués d'un corps strictement convexe roulant [10] sur le plan. Nous présentons au chapitre 4 une implémentation numérique de la méthode de continuation appliquée au problème de roulement sur le plan.

Remarque 2.15. Bien que la convergence théorique de la méthode de continuation (équivalente à l'existence globale de la solution de l'équation de relèvement) soit difficile à obtenir en général, l'implémentation numérique de cette méthode (ou ses variantes), c'est à dire la résolution de l'équation de relèvement (2.33) par des schémas numériques, fournit souvent rapidement une commande approchée très satisfaisante pour des exemples concrets.

2.4 Autres méthodes

Il existe beaucoup d'autres approches pour résoudre le problème de planification de mouvement. Nous avons choisi d'en présenter deux : celles basées sur la platitude et le contrôle optimal. Signalons tout de suite que ces deux approches, comme la méthode de continuation, ne sont pas spécifiques aux systèmes non-holonomes et qu'elles sont a priori applicables aux systèmes de contrôle de la forme

$$\dot{x} = f(x,u), \tag{2.35}$$

où f est une application de classe C^1 de $\mathbb{R}^n \times \mathbb{R}^m$ à valeurs dans $\mathbb{R}^n$.

Cependant, des limitations de diverse nature font qu'aucune des deux approches parvient à résoudre de façon générale le problème de planification pour des systèmes nonholonomes. La présentation qui va suivre est volontairement succincte et le lecteur est invité à consulter les références indiquées pour des exposés plus détaillés.

2.4.1 Platitude

La méthode de commande par platitude a été développée par M. Fliess, J. Lévine, P. Martin et P. Rouchon au début des années 90. L'idée principale de cette méthode repose sur une paramétrisation explicite des trajectoires d'un système de la forme (2.35).

Définition 2.2 (Système plat)**.** Un système commandé de la forme (2.35) est dit *plat* s'il existe un entier naturel r et des applicationd $h : \mathbb{R}^n \times (\mathbb{R}^m)^{r+1} \to \mathbb{R}^m$, $\Psi : (\mathbb{R}^m)^r \to \mathbb{R}^n$ et $\psi : (\mathbb{R}^m)^{r+1} \to \mathbb{R}^m$ tels que, si on définit

$$y := h(x,u,\dot{u},\ldots,u^{(r)}),$$

alors les commandes et les trajectoires du système sont données par

$$\begin{aligned} u &= \psi(y,\dot{y},\ldots,y^{(r)}), \\ x &= \Phi(y,\dot{y},\ldots,y^{(r-1)}). \end{aligned}$$

10. Rouler signifie sans glisser ni pivoter.

La variable y est appelée *sortie plate* pour le système.

Cette définition revient à dire que si un système est plat, alors on peut résumer tout son comportement dynamique par le comportement de sa sortie plate. De façon équivalente, si un système est plat, on peut alors paramétrer par la sortie plate et un nombre fini de ses dérivées successives toutes les trajectoires et les commandes. On dit qu'il y a une correspodance bi-univoque entre les trajectoires d'un système plat et celles des sorties plates.

Grâce à cette correspondance bi-univoque, il est facile de faire la planification de mouvement pour des systèmes plats. Supposons que l'on cherche à amener un système plat (dont on a trouvé une sortie plate y) de x_0 à x_1 dans l'intervalle $[0, T]$. Par définition, on a

$$x_0 = \Psi(y(0), \dot{y}(0), \ldots, y^{(r)}(0)), \tag{2.36}$$
$$x_1 = \Psi(y(T), \dot{y}(T), \ldots, y^{(r)}(T)). \tag{2.37}$$

Il suffit ensuite de trouver une courbe dans l'espace des sorties plates $t \to y(t)$ définie sur $[0, T]$ satisfaisant le système d'équations donné par (2.36) et (2.37), ce qui peut être réalisé par de simple interpolations polynomiales. Par construction, la commande $u : t \to u(t)$ définie par

$$u(t) := \psi(y(t), \dot{y}(t), \ldots, \dot{y}^{(r+1)}),$$

amène le système de x_0 à x_1.

Cette approche fournit une méthode exacte et efficace de planification pour les systèmes plats. La notion de la platitude peut aussi s'étendre aux systèmes en dimension infinie. Le lecteur pourra consulter [37] pour un exposé plus détaillé. Un catalogue de systèmes plats (en dimensions finie et infinie) est présenté dans [71].

Pour les systèmes généraux, la principale limitation de cette approche est due au fait qu'il n'existe pas de méthode générale permettant de trouver les sorties plates. Pour les systèmes non-holonomes, il semble que la platitude n'est pas une propriété générique pour les systèmes non-holonomes. Par exemple, dans le cas où $m = 2$, sous des hypothèses de régularités adéquates, on peut montrer que les systèmes plats sont équivalents par bouclage statique aux systèmes chainés (cf. [70]). C'est donc une classe très restreinte de systèmes non-holonomes.

2.4.2 Contrôle optimal

La méthode la mieux formulée pour résoudre le problème de planification est probablement celle qui utilise le contrôle optimal. L'idée est d'associer un *coût* à chaque trajectoire admissible [11] et de chercher une trajectoire admissible qui minimise ce coût. En résolvant ce problème d'optimisation, nous obtenons une trajectoire (en fait, une commande qui permet de générer cette trajectoire) qui relie le point initial au point final (le problème de planification est donc résolu), tout en optimisant le coût. On distingue en général deux types de méthodes numériques pour résoudre ce problème : les méthodes directes et les méthodes indirectes. Les méthodes directes consistent à discrétiser la fonction coût, l'espace d'état et l'espace de contrôle, en réduisant le problème à un problème d'optimisation non linéaire sous contraintes. On y trouve en particulier une approche par la programmation dynamique qui conduit à la résolution de l'équation de Hamilton-Jacobi-Bellman. Les méthodes indirectes consistent à résoudre numériquement, par une méthode de tir, un problème aux valeurs limites obtenu par l'application du principe du maximum de Pontryagin. Un exposé détaillé

11. Les trajectoires admissibles dans ce contexte sont les trajectoires qui relient le point initial et le point final tout en respectant les diverses contraintes (contraintes sur la dynamique, contraintes sur l'ensemble des contrôles etc.).

avec des exemples d'implémentation numérique peut être trouvé dans [104, Chapitre 9]. Une discussion particulièrement intéressante sur le choix de ces méthodes numériques en pratique se trouve à la section 9.3 de cette même référence.

Les principales difficultés de l'approche du contrôle optimal sont :

(i) la convergence vers des minima locaux qui ne garantissent pas l'admissibilité des trajectoires trouvées [12] ;

(ii) la limitation sur la dimension de l'espace d'états due à la discrétisation.

Remarque 2.16. Le point (i) est un défaut plutôt intrinsèque, bien connu en optimisation. Il n'existe pas de méthodes générales permettant de le résoudre de façon satisfaisante dans le cadre du contrôle optimal. Notons que l'approche HJB garantit la convergence vers le minimum global, mais elle est grandement limité par le point (ii) à cause de la croissance exponentielle de la complexité, un défaut commun à toute approche par la programmation dynamique en optimisation.

Une liste de références détaillée sur l'utilisation du contrôle optimal pour résoudre le problème de planification est disponible à [21, Section 13.1.2]. J.-P. Gauthier, B.Ł Jakubczyk et V. Zakalyukin ont récemment obtenu, en supposant que le système non-holonome est *fortement* Lie-généré, un nouvel algorithme de planification par l'approche du contrôle optimal en utilisant des commandes sinusoïdales (cf. [38]). Cette méthode est à comparer avec celle de Liu-Sussmann présentée au paragraphe 2.2.3 et la nôtre présentée au chapitre 3.

2.5 Commentaires sur les méthodes existantes

Pour comparer les principales caractéristiques des méthodes existentes, nous introduisons quelques critères que nous regroupons sous le nom de *procédure complète* : un algorithme de planification de mouvements pour les systèmes non-holonomes est une procédure complète si toutes les propriétés suivantes sont satisfaites :

1. Généralité : l'algorithme doit fonctionner pour tous les systèmes non-holonomes, sans restriction a priori sur la structure du système.
2. Globalité : pour toute paire de points (p, q) dans l'espace d'états Ω, l'algorithme doit produire une commande qui amène le système du point p à un point $\tilde{q} \in \Omega$, arbitrairement proche du point q [13].
3. Preuve théorique : une preuve mathématique garantissant le point 2 doit être fournie.
4. Utilisabitlité pratique : l'algorithme doit
   (a) fonctionner sans restriction a priori sur la dimension du système ;
   (b) être robuste par rapport à la dynamique du système ;
   (c) produire des trajectoires régulières (sans cusps ni trop d'oscillations) ;
   (d) être généralisable au cas avec obstables dans Ω.

Nous récapitulons dans le tableau suivant les principales caractéristiques des méthodes existantes par rapport à ces quatre critère.

En conclusion, *aucune* des méthodes existantes n'est une procédure complète. Nous présentons chapitre 3 *une première procédure complète* pour résoudre le problème de la planification de mouvements dans le cadre des systèmes non-holonomes généraux.

12. Pour les méthodes directes, ce problème est lié à la discrétisation. Pour les méthodes indirectes, le principe du maximum n'est qu'une condition nécessaire d'optimalité.

13. Le cas du contrôle exact est inclus.

TABLE 2.1 – Tableau de comparaison

	Généralité	Globalité	Preuve	Utilisabilité
Murray-Sastry	Non	Oui	Oui	Oui
Lafferriere-Sussmann	Oui	Oui	Oui	Non
Liu-Sussmann	Oui	Oui	Oui	Non
Continuation	Oui	Oui	Non	Oui
Platitude	Non	Oui	Oui	Oui
Contrôle optimal	Oui	Oui	Non	Non

Chapitre 3

A Global Steering Method for Nonholonomic Systems

Le contenu de ce chapitre fait l'objet d'un article en collaboration avec Y. Chitour et F. Jean, soumis au *Journal of Differential Equations*.

Sommaire

3.1 Introduction

Nonholonomic systems have been attracting the attention of the scientific community for several years, due to the theoretical challenges they offer and the numerous important applications they cover. From the point of view of control theory, a nonholonomic system is a driftless control-affine system and is written as

$$(\Sigma) \quad \dot{x} = \sum_{i=1}^{m} u_i X_i(x), \ x \in \Omega, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m, \tag{3.1}$$

where Ω is an open connected subset of $\mathbb{R}^n$, and $X_1, \dots, X_m$ are C^∞ vector fields on Ω. Admissible inputs are $\mathbb{R}^m$-valued measurable functions $u(\cdot)$ defined on some interval $[0, T]$

and a trajectory of (Σ), corresponding to some $x_0 \in \Omega$ and to an admissible input $u(\cdot)$, is the (maximal) solution $x(\cdot)$ in Ω of the Cauchy problem defined by $\dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t))$, $t \in [0, T]$, and $x(0) = x_0$.

In this paper, we address the *motion planning problem* (MPP for short) for (Σ), namely determine a procedure which associates with every pair of points $(p, q) \in \Omega \times \Omega$ an admissible input $u(\cdot)$ defined on some interval $[0, T]$, such that the corresponding trajectory of (Σ) starting from p at $t = 0$ reaches q at $t = T$. As for the existence of a solution to MPP, this is equivalent to the complete controllability of (Σ). After the works of Chow and Rashevsky in the thirties [30, 82], and that of Sussmann and Stefan in the seventies [93, 92], the issue of complete controllability for nonholomic systems is well-understood and it is usually guaranteed by assuming that the Lie Algebraic Rank Condition (also known as the Hörmander condition) holds for (Σ). This easily checkable condition is not only sufficient for complete controllability but also necessary when the vector fields are analytic. From a practical viewpoint, assuming the LARC is, in a sense, the minimal requirement to ensure complete controllability for (Σ) and this is what we will do for all the control systems considered hereafter.

As for the construction of the solutions of the MPP, we present, in this paper, a complete procedure solving the MPP for a nonholonomic system subject to the sole LARC. By "complete procedure", we mean that the following properties must be guaranteed by the proposed procedure.

1. Global character of the algorithm : for every pair of points (p, q) in Ω, the algorithm must produce a steering control. (Note that the core of many algorithms consists in a local procedure and turning the latter into a global one is not always a trivial issue.)
2. Proof of convergence of the algorithm.
3. Regarding numerical implementations, no prohibitive limitation on the state dimension n.
4. Usefulness for practical applications, e.g., robustness with respect to the dynamics, "nice" trajectories produced by the algorithm, (no cusps neither large oscillations), and possibility of localizing the algorithm in order to handle obstacles (i.e., reducing the working space Ω to any smaller open and connected subset of $\mathbb{R}^n$).

There exist of course several algorithms addressing the MPP in different contexts but most of them fail to verify all the aforementioned properties.

At first, in the case of specific classes of driftless nonholonomic systems (i.e. where more is known than the sole LARC), effective techniques have been proposed, among which a Lie bracket method for steering nilpotentizable systems (see [62] and [63]), sinusoidal controls for chained-form systems (see [76]), averaging techniques for left-invariant systems defined on a Lie group (see [64, 22]), and a trajectory generation method for flat systems (see [37]). Depending on the applications, these methods turn out to be extremely efficient, especially when the system to be steered is shown to be flat with an explicit flat output.

However, the class of systems considered previously is rather restrictive : for 2-input nonholonomic systems (i.e. $m = 2$), under suitable regularity assumptions, a flat system admits a feedback chained-form transformation (cf. [74, 70]) and thus is exactly nilpotentizable ; on the other hand, when the dimension of the state space is large enough, exact nilpotentizability is clearly a non generic property among 2-input nonholonomic systems. Moreover, there exist standard nonholonomic systems whose kinematic model does not fall into any of the aforementioned categories. For instance, mobile robots with more than one trailer cannot be transformed in chained-form unless each trailer is hinged to the midpoint of the previous wheel axle, an unusual situation in real vehicles. Another similar example is the rolling-body

problem : even the simplest model in this category, the so-called plate-ball system, does not allow any chained-form transformation and is not flat.

Regarding general nonholonomic systems, various steering techniques have been proposed in the literature, and we only mention three of them : the iterative method, the generic loop method, and the continuation method. The first one, introduced in [62] and improved in [63], is an approximation procedure which is exact for nilpotent systems. This method is proved to be convergent with the sole assumption of the LARC and actually meets most of the requirements to be a complete procedure in the sense defined previously. However, either the resulting trajectories in [63] contain a large number of cusps (exponential with respect to the degree of nonholonomy), or the computation of the steering control in [62] requires the inversion of a system of algebraic equations. The latter turns out to be numerically intractable as soon as the dimension of the state is larger than six. Let us also mention a less important limitation for practical use. The iterative method described in [62, 63] makes use of several nonlinear changes of coordinates, which must be performed by numerical integration of ODEs at each step of the iterative method, thus leading to spurious on-line computations.

The generic loop method, presented in [90], is based on a local deformation procedure and requires an a priori estimate of some “critical distance" which is, in general, an unknown parameter in practice. That fact translates into a severe drawback for constructing a globally valid algorithm. The continuation method of [97] and [25] belongs to the class of Newton-type methods. Proving its convergence amounts to show the global existence for the solution of a non linear differential equation, which relies on handling the abnormal extremals associated to the control system. That latter issue turns out to be a hard one, see [72, 27, 28] for instance. This is why, in the current state of knowledge, the continuation method can be proved to converge only under restrictive assumptions (see [23, 24, 29]).

The algorithm considered in the present paper takes as starting point the globally convergent algorithm for steering *regular* nonholonomic systems discussed in [54]. As the iterative method of [62, 63], that algorithm can be casted in the realm of nonlinear geometric control and roughly works as follows : one first solves the motion planning problem for a control system which is nilpotent and “approximates" system (3.1) in a suitable sense ; then, one applies the resulting input $\hat{u}$ to (3.1) and iterates the procedure from the current point. If we use $\hat{x}(t, a, \hat{u})$, $t \in [0, T]$ to denote a trajectory of the “approximate" control system starting from a, a local version of this algorithm is summarized below, where d is an appropriate distance (to be defined in the next section) and e is a fixed positive real number.

Algorithm 4 Local Steering Algorithm

Require: x_0, x_1, e
  $k = 0$;
  $x^k = x_0$;
  while $d(x^k, x_1) > e$ **do**
    Compute $\hat{u}^k$ such that $x_1 = \hat{x}(T, x^k, \hat{u}^k)$;
    $x^{k+1} = \text{AppSteer}\ (x^k, x_1) := x(T, x^k, \hat{u}^k)$;
    $k = k + 1$;

We note that Algorithm 4 converges *locally* provided that the function AppSteer is *locally contractive* with respect to the distance d, i.e., for $x_1 \in \Omega$, there exists $\varepsilon_{x_1} > 0$ and $c_{x_1} < 1$ such that

$$d(\text{AppSteer}(x, x_1), x_1) \leq c_{x_1} d(x_1, x), \tag{3.2}$$

for $x \in \Omega$ and $d(x_1, x) < \varepsilon_{x_1}$.

Assume now that we have a *uniformly locally contractive* function AppSteer on a connected compact set $K \subset \Omega$, i.e. there exists $\varepsilon_K > 0$ and $c_K \in (0,1) < 1$ such that

$$d(\text{AppSteer}(x, x_1), x_1) \leq c_K d(x_1, x), \tag{3.3}$$

for $x,\ x_1 \in K$ and $d(x_1, x) < \varepsilon_K$.

Based on the local algorithm, a global approximate steering algorithm on K can be built along the line of the following idea (a similar procedure is proposed in [63]) : consider a parameterized path $\gamma \subset K$ connecting x_0 to x_1. Then pick a finite sequence of intermediate goals $\{x_0^d = x_0, x_1^d, \ldots, x_j^d = x_1\}$ on γ such that $d(x_{i-1}^d, x_i^d) < \varepsilon_K/2$, $i = 0, \ldots, j$. One can prove that the iterated application of a uniformly locally contractive AppSteer(x^{i-1}, x_i^d) from the current state to the next subgoal (having set $x_i^d = x_1$, for $i \geq j$) yields a sequence x^i converging to x_1.

To turn the above idea into a practically efficient algorithm, three issues must be successfully addressed :

(P-1) Construct a *uniformly* locally contractive local approximate steering method.

(P-2) The "approximate" control $\hat{u}^k$ must be *exact* for steering the "approximate system" from the current point x^k to the final point x_1. As this computation occurs at each iteration, it must be performed in a reasonable time.

(P-3) Since the knowledge of the "critical distance" ε_K is not available in practice, the algorithm should achieve global convergence without explicit knowledge of ε_K.

Issues (P-1) and (P-3) are solved in [54] under the assumption that the control system is *regular* (cf. Definition 3.9 below). The solution proposed therein relies on the understanding of the geometry defined by the nonholonomic system (cf. [12]). That geometry is a sub-Riemannian one and it endows the working space Ω with a sub-Riemannian metric d for which the aforementioned function AppSteer is contractive. Moreover, the approximation of the original system adapted to the motion planning turns out to be the approximation at the first order with respect to d (cf. [54]). However, the regularity assumption for the control system is rather restrictive : in general, nonholonomic systems do exhibit singularities (cf. [107]). A solution also exists in the case of a non regular control system (cf. [106]), but only when the state dimension n is less than or equal to 5. In the present paper, we completely remove the regularity assumption and solve Issues (P-1) and (P-3) for every nonholonomic control system. The solution is based on an explicit desingularization procedure : adding new variables (thus augmenting the dimension of the state space), we construct a "lifted" control system which is regular and whose projection is the original control system. The solution of Issue (P-1) described in [54] can thus be applied to the "lifted" control system, as well as the globally convergent motion planning algorithm solving Issue (P-3) proposed therein. Note that other desingularization procedures already exist [11, 39, 52, 84], but we insist on the fact that the one we propose here involves only *explicit* polynomial transformations. It numerically translates to the fact that these changes of variables can be performed off-line once each local procedure is identified.

As regards Issue (P-2), several algorithms were proposed for computing $\hat{u}$, i.e. for controlling nilpotent systems. In [62], the authors make use of piecewise constant controls and obtain smooth controls by imposing some special parameterization (namely by requiring the control system to stop during the control process). In that case, the smoothness of the inputs is recovered by using a reparameterization of the time, which cannot prevent in general the occurrence of cusps or corners for the corresponding trajectories. However, smoothness of the *trajectories* is generally mandatory for robotic applications. Therefore, the method proposed in [62] is not adapted to such applications. In [63], the proposed controls are polynomial (in time), but an algebraic system must be inverted in order to access to these inputs. The size

and the degree of this algebraic system increase exponentially with respect to the dimension of state space, and there does not exist a general efficient exact method to solve it. Even the existence of solutions is a non trivial issue. Furthermore, the methods [62] and [63] both make use of exponential coordinates which are not explicit and thus require in general numerical integrations of nonlinear differential equations. That prevents the use of these methods in an iterative scheme such as Algorithm 4. Let us also mention the path approximation method by Liu and Sussmann [68], which uses unbounded sequences of sinusoids. Even though this method bears similar theoretical aspects with our method (see especially the argument strategy in order to prove Lemma 3.25, which is borrowed from [68]), it is not adapted from a numerical point of view to the motion planning issue since it relies on a limit process of highly oscillating inputs. In the present paper, we present an *exact* steering algorithm for general nilpotent systems is provided, which uses sinusoidal inputs and which can be applied for controlling the approximate (nilpotent) system used in [54]. Our method generalizes the one proposed in [76] for controlling chained-form systems, which is briefly recalled next : after having brought the system under a "canonical" form, the authors of [76] proceed by controlling component after component by using, for each component, two sinusoids with suitably chosen frequencies. In the present paper, we show for general nilpotent systems that, with more frequencies for each component, one can steer an arbitrary component independently on the other components. We are also able to construct inputs which give rise to C^1-trajectories.

We now describe in a condensed manner the global motion planning strategy developed in this paper. The latter is presented as an algorithmic procedure associated with a given nonholonomic system (Σ) defined on $\Omega \subset \mathbb{R}^n$. The required inputs are initial and final points x^{initial} and x^{final} belonging to Ω, a tolerance $e > 0$, and a compact convex set $K \subset \Omega$ (of appropriate size) equal to the closure of its interior which is a neighborhood of both x^{initial} and x^{final}. For instance, K can be chosen to be a large enough compact tubular neighborhood constructed around a curve joining x^{initial} and x^{final}. The global steering method is summarized in Algorithm 5.

The paper is devoted to the construction of the various steps of this algorithm. We will also show that each of these steps is conceived so that the overall construction is a complete procedure in the sense defined previously. In particular, the convergence issue is addressed in the following theorem.

Theorem 3.1. *Let (Σ) be a nonholonomic system on $\Omega \subset \mathbb{R}^n$ satisfying the LARC. For every $e > 0$, every connected compact set K which is equal to the closure of its interior, and every pair of points $(x^{\text{initial}}, x^{\text{final}})$ in the interior of K, Algorithm 5 steers, in a finite number of steps, the control system (Σ) from x^{initial} to a point $x \in K$ such that $d(x, x^{\text{final}}) < e$.*

Before providing the structure of the paper, we mention possible extensions of our algorithm. The first one concerns the working space Ω. Since it is an arbitrary open connected set of $\mathbb{R}^n$, one can extend the algorithm to the case where the working space is a smooth connected manifold of finite dimension. From a numerical point of view, there would be the additional burden of computing the charts. A second extension deals with the stabilization issue. Indeed, at the heart of the algorithm lies an iterative procedure such as Algorithm 4, which can be easily adapted for stabilization tasks (cf. [79]). Another possible generalization takes advantage of devising from our algorithm a globally regular input, one can then address the motion planning of dynamical extensions of the nonholonomic control systems considered in the present paper. Finally, let us point out the modular nature of Algorithm 5 : one can

Algorithm 5 Global Approximate Steering Algorithm : Global($x^{\text{initial}}, x^{\text{final}}, e, K$)

1: Build a decomposition of K into a finite number of compacts sets $\mathcal{V}^c_{\mathcal{J}_i}$, with $i = 1, \dots, M$ (Section 3.3.3).
2: Construct the connectedness graph $\mathsf{G} := (\mathsf{N},\ \mathsf{E})$ associated with this decomposition and choose a simple path $\mathsf{p} := \{j_0, j_1 \dots, j_{\bar{M}}\}$ in G such that $x^{\text{initial}} \in \mathcal{V}^c_{\mathcal{J}_{j_0}}$ and $x^{\text{final}} \in \mathcal{V}^c_{\mathcal{J}_{j_{\bar{M}}}}$ (Section 3.3.3).
3: Choose a sequence $(x^i)_{i=1,\dots,\bar{M}-1}$ such that $x^i \in \mathcal{V}^c_{\mathcal{J}_{j_i}} \cap \mathcal{V}^c_{\mathcal{J}_{j_{i+1}}}$.
4: Set $x := x^{\text{initial}}$.
5: **for** $i = 1, \dots, \bar{M} - 1$ **do**
6: Apply the Desingularization Algorithm at $a := x^i$ with $\mathcal{J} := \mathcal{J}_i$ (Section 3.3.3).
{the output is an m-tuple of vector fields ξ on $\mathcal{V}_{\mathcal{J}_i} \times \mathbb{R}^{\widetilde{n}}$ which is free up to step r.}
7: Let AppSteer be the LAS method associated to the approximation $\mathcal{A}^\xi$ of ξ on $\mathcal{V}_{\mathcal{J}_i} \times \mathbb{R}^{\widetilde{n}}$ defined in Section 3.4.1 and to its steering law $\text{Exact}_{m,r}$ constructed in Section 3.5.3).
8: Set $\widetilde{x}_0 := (x, 0)$, $\widetilde{x}_1 := (x^i, 0)$, and $\mathcal{V}^c := \mathcal{V}^c_{\mathcal{J}_i} \times \overline{B}_R(0)$ with $R > 0$ large enough.
9: Apply GlobalFree($\widetilde{x}_0, \widetilde{x}_1, e, \mathcal{V}^c$, AppSteer) to ξ (Section 3.4.2).
{the algorithm stops at a point $\widetilde{x}$ which is e-close to $\widetilde{x}_1$;}
10: **return** $x := \pi(\widetilde{x})$.
{$\pi : \mathcal{V}_{\mathcal{J}_i} \times \mathbb{R}^{\widetilde{n}} \to \mathcal{V}_{\mathcal{J}_i}$ is the canonical projection.}

propose other approaches to obtain uniformly contractive local methods (other desingularization methods or different ways of dealing with singular points), or replace $\text{Exact}_{m,r}(\cdot)$ by more efficient control strategies for general nilpotent systems.

The paper is organized as follows. In Chapter 3.2, we define properly the notion of first order approximation. We then propose, in Chapter 3.3, a purely polynomial desingularization procedure based on a *lifting* method. In Chapter 3.4, we describe in detail the globally convergent steering algorithm given in [54] for regular systems together with a proof of convergence. In Chapter 3.5, we present an exact steering procedure for general nilpotent systems using sinusoids, and we gather, in Appendix 3.6, the proof of Theorem 3.1 and some additional comments.

3.2 Notations and Definitions

Let n and m be two positive integers. Let Ω and $VF(\Omega)$ be respectively an open connected subset of $\mathbb{R}^n$ and the set of C^∞ vector fields on Ω. Consider m vector fields $X_1, \dots, X_m$ of $VF(\Omega)$, and the associated driftless control-affine nonholonomic system given by

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x),\ \ x \in \Omega, \tag{3.4}$$

where $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$ and the input $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ is an integrable vector-valued function defined on $[0, T]$, with T a fixed positive real number.

We also assume that (4.5) is *complete*, i.e., for every $a \in \Omega$ and input $u(\cdot)$, the Cauchy problem defined by (4.5) starting from a at $t = 0$ and corresponding to $u(\cdot)$ admits a unique (absolutely continuous) solution $x(\cdot, a, u)$ defined on $[0, T]$ and called the trajectory of (4.5)

starting from a at $t = 0$ and corresponding to the input $u(\cdot)$. A point $x \in \Omega$ is said to be *accessible* from a if there exists an input $u : [0, T] \to \mathbb{R}^m$ and a time $t \in [0, T]$ such that $x = x(t, a, u)$. Then, System (4.5) is said to be completely controllable if any two points in Ω are accessible one from each other (see [94]).

We next provide a classical condition ensuring that System (4.5) is controllable. We first need the following definition.

Définition 3.1 (*Lie Algebraic Rank Condition (LARC)*)**.** Let $L(X)$ be the Lie algebra generated by the vectors fields $X_1, \ldots, X_m$ and $L(x)$ be the linear subspace of $\mathbb{R}^n$ equal to the evaluation of $L(X)$ at every point $x \in \Omega$ (see [11]). If $L(x) = \mathbb{R}^n$, we say that the *Lie Algebraic Rank Condition* (LARC for short) is verified at $x \in \Omega$. If this is the case at every point $x \in \Omega$, we say that System (4.5) satisfies the LARC.

Chow's Theorem essentially asserts that, if System (4.5) satisfies the LARC then it is completely controllable (cf. [30]).

Remark 3.1. For the sake of clarity, we assume through this paper that the control set is equal to $\mathbb{R}^m$. However, it is well-known that Chow's theorem only requires that the convex hull of the control set contains a neighborhood of the origin in $\mathbb{R}^m$ (see for instance [55, Chapter 4, Theorem 2]). We will explain later how we can adapt our method to the case with constraints on the control set (see Section 3.6.2). Moreover, it is worth recalling that complete controllability for (Σ) does not imply that LARC holds true for (Σ) if the vector fields $X_1, \ldots, X_m$ are only smooth, but this is the case if $X_1, \ldots, X_m$ are analytic (cf. [2, Chapter 5]).

Throughout this paper, we will only consider driftless control-affine nonholonomic systems of the type (4.5) verifying the LARC, and thus completely controllable. In that context, the *motion planning problem* will be defined as follows : find a procedure which furnishes, for every two points x_0, $x_1 \in \Omega$, an input u steering (4.5) from x_0 to x_1, i.e., $x(T, x_0, u) = x_1$.

Our solution to this problem relies heavily on the underlying geometry, which is a *sub-Riemannian geometry*. We provide in Section 3.2.1 the useful definitions and refer the reader to [11] and [73] for more details. We then introduce in Section 3.2.2 a notion of approximate steering method related to this geometry.

3.2.1 Basic facts on sub-Riemannian geometry

3.2.1.1 Sub-Riemannian distance and Nonholonomic order

Définition 3.2 (*Length of an input*)**.** The *length of an input* u is defined by

$$\ell(u) = \int_0^T \sqrt{u_1^2(t) + \cdots + u_m^2(t)}dt,$$

and the *length of a trajectory* $x(\cdot, a, u)$ is defined by

$$\ell(x(\cdot, a, u)) := \ell(u).$$

The appropriate notion of distance associated with the control system (4.5) and closely related to the notion of accessibility is that of *sub-Riemannian distance*, also called *control distance*.

Définition 3.3 (*Sub-Riemannian distance*)**.** The vector fields $X_1, \ldots, X_m$ induce a function d on Ω, defined by

$$d(x_1, x_2) := \inf_u \ell(x(\cdot, x_1, u)), \text{ for every points } x_1, x_2 \text{ in } \Omega, \tag{3.5}$$

where the infimum is taken over all the inputs u such that $x(\cdot, x_1, u)$ is defined on $[0, T]$ and $x(T, x_1, u) = x_2$. We say that the function d is the *sub-Riemannian distance* associated with $X_1, \ldots, X_m$.

Remark 3.2. The function d defined above is a *distance* in the usual sense, i.e., it verifies (i) $d(x_1, x_2) \geq 0$ and $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$; (ii) symmetry : $d(x_1, x_2) = d(x_2, x_1)$; (iii) triangular inequality : $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$. Notice that one always has $d(x_1, x_2) < \infty$ since the control system is assumed to be completely controllable.

Définition 3.4 (*Nonholonomic derivatives of a function*)**.** If

$$f : \Omega \to \mathbb{R}^n$$

is a smooth function, the *first-order nonholonomic derivatives* of f are the Lie derivatives $X_i f$ of f along X_i, $i = 1, \ldots, m$. Similarly, $X_i(X_j f)$, $i, j = 1, \ldots, m$, are called the *second-order nonholonomic derivatives* of f, and more generally, $X_{i_1} \cdots X_{i_k} f$, $i_1, \ldots, i_k \in \{1, \ldots, m\}$ are the k^{th}*-order nonholonomic derivatives* of f, where k is any positive integer.

Proposition 3.2 ([11, Proposition 4.10, page 34])**.** *Let s be a non-negative integer. For a smooth function f defined near $a \in \Omega$, the following conditions are equivalent :*

(i) *$f(x) = O(d^s(x, a))$ for x in a neighborhood of a ;*

(ii) *all the nonholonomic derivatives of order $\leq s - 1$ of f vanish at a.*

Définition 3.5 (*Nonholonomic order of a function*)**.** Let s and f be respectively a non-negative integer and a smooth real-valued function defined on Ω. If Condition (i) or (ii) of Proposition 3.2 holds, we say that f is of *order* $\geq s$ at $a \in \Omega$. If f is of order $\geq s$ but not of order $\geq s + 1$ at a, we say that f is of *order* s at a. The order of f at a will be denoted by $\mathrm{ord}_a(f)$.

Définition 3.6 (*Nonholonomic order of a vector field*)**.** Let q be an integer. A vector field $Y \in VF(\Omega)$ is of *order* $\geq q$ at $a \in \Omega$ if, for every non-negative integer s and every smooth function f of order s at a, the Lie derivative Yf is of order $\geq q + s$ at a. If Y is of order $\geq q$ but not $\geq q + 1$, it is of *order* q at a. The order of Y at a will be denoted by $\mathrm{ord}_a(Y)$.

Définition 3.7 (*Nonholonomic first order approximation at a*)**.** An m-tuple

$$\widehat{X}^a := \{\widehat{X}_1^a, \ldots, \widehat{X}_m^a\},$$

defined on $B(a, \rho_a) := \{x \in \Omega,\ d(x, a) \leq \rho_a\}$ with $\rho_a > 0$, is said to be a *nonholonomic first order approximation of* $X := \{X_1, \ldots, X_m\}$ *at* $a \in \Omega$, if the vector fields $X_i - \widehat{X}_i^a$, for $i = 1, \ldots, m$, are of order ≥ 0 at a. The positive number ρ_a is called *the approximate radius at* a.

Remark 3.3. As a consequence of Definition 3.7, one gets that the nonholonomic order at a defined by the vector fields $\widehat{X}_1^a, \ldots, \widehat{X}_m^a$ coincides with the one defined by $X_1, \ldots, X_m$.

3.2.1.2 Privileged coordinates

The changes of coordinates take an important place in this paper, whether it is to estimate the sub-Riemannian distance, or to compute the order of functions and vector fields, or to transform a control system into a normal form. To avoid heavy notations, we will need some conventions and simplifications that we fix now for the rest of the paper.

A point in $\Omega \subset \mathbb{R}^n$ is denoted by $x = (x_1, \dots, x_n)$ and the canonical basis of $\mathbb{R}^n$ by $(\partial_{x_1}, \dots, \partial_{x_n})$. Even though x is a point, we will sometimes refer to $(x_1, \dots, x_n)$ as the original coordinates. A *system of local coordinates* $y = (y_1, \dots, y_n)$ at a point $a \in \Omega$ is defined as a diffeomorphism φ between an open neighborhood $N_a \subset \Omega$ of a and an open neighborhood $N_{\varphi(a)} \subset \mathbb{R}^n$ of $\varphi(a)$,

$$\varphi : x \mapsto y = (y_1, \dots, y_n).$$

If the diffeomorphism φ is defined on Ω, then $y = (y_1, \dots, y_n)$ is said to be a *system of global coordinates* on Ω. A system of global coordinates is said to be *affine* (resp. *linear*) if the corresponding diffeomorphism φ is affine (resp. linear). If f is a function defined on N_a, the function $f \circ \varphi^{-1}$ defined on $N_{\varphi(a)}$ will be called f *(expressed) in coordinates* $(y_1, \dots, y_n)$. If $X \in VF(\Omega)$ is a vector field, the push-forward $\varphi_* X = d\varphi \circ X \circ \varphi^{-1} \in VF(N_{\varphi(a)})$ will be called X *(expressed) in coordinates* $(y_1, \dots, y_n)$.

For the sake of simplicity, we will in general not introduce the notation φ and, with a slight abuse of the notation, replace it by y. Thus we write $y(x)$ or $(y_1(x), \dots, y_n(x))$ instead of $\varphi(x)$. The function $f \circ \varphi^{-1}$ will be denoted by $f(y)$, and the vector field $\varphi_* X$ by $X(y)$ or $y_* X$. The values at a point $\bar{y} \in N_{\varphi(a)}$ will be denoted respectively by $f(y)_{|y=\bar{y}}$ and $X(y)_{|y=\bar{y}}$.

A special class of coordinates, called *privileged coordinates* (cf. [73, Chapter 2]) and defined below, turns out to be a useful tool to compute the order of functions and vector fields, and to estimate the sub-Riemannian distance d.

We will use $L^s(X)$ to denote the Lie sub-algebra of elements of length (cf. Definitions 3.20 and 3.23) not greater than $s \in \mathbb{N}$. Take $x \in \Omega$ and let $L^s(x)$ be the vector space generated by the values at x of elements belonging to $L^s(X)$. Since System (4.5) verifies the LARC at every point $x \in \Omega$, there exists a smallest integer $r := r(x)$ such that $\dim L^r(x) = n$. This integer is called the *degree of nonholonomy* at x.

Définition 3.8 (*Growth vector*)**.** For $a \in \Omega$, let $n_s(a) := \dim L^s(a)$, $s = 1, \dots, r$. The sequence

$$(n_1(a), \dots, n_r(a))$$

is the *growth vector* of X at a.

Définition 3.9 (*Regular and singular points*)**.** A point $a \in \Omega$ is said to be *regular* if the growth vector remains constant in a neighborhood of a and, otherwise, a is said to be *singular*. The nonholonomic System (4.5) (or the m-tuple X) is said to be *regular* if every point in Ω is regular.

Note that regular points form an open and dense set in Ω.

Définition 3.10 (*Weight*)**.** For $a \in \Omega$ and $j = 1, \dots, n$, let $w_j := w_j(a)$ be the integer defined by setting $w_j := s$ if $n_{s-1} < j \leq n_s$, with $n_s := n_s(a)$ and $n_0 := 0$. The integers w_j, for $j = 1, \dots, n$ are called the *weight at* a.

Remark 3.4. The meaning of Definition 3.10 can be understood in another way. Choose first some vector fields $W_1, \dots, W_{n_1}$ in $L^1(X)$ such that $W_1(a), \dots, W_{n_1}(a)$ form a basis of $L^1(a)$. Choose then other vectors fields $W_{n_1+1}, \dots, W_{n_2}$ in $L^2(X)$ such that $W_1(a), \dots, W_{n_2}(a)$ form

a basis of $L^2(a)$ and, for every positive integer s, choose $W_{n_{s-1}+1}, \dots, W_{n_s}$ in $L^s(X)$ such that $W_1(a), \dots, W_{n_s}(a)$ form a basis of $L^s(a)$. We obtain in this way a sequence of vector fields $W_1, \dots, W_n$ such that

$$\begin{cases} W_1(a), \dots, W_n(a) \text{ is a basis of } \mathbb{R}^n, \\ W_i \in L^{w_i}(X), i = 1, \dots, n. \end{cases} \tag{3.6}$$

A sequence of vector fields verifying Eq. (3.6) is called an *adapted frame at a*. The word "adapted" means "adapted to the flag $L^1(a) \subset L^2(a) \subset \cdots \subset L^r(a) = \mathbb{R}^n$", since the values at a of an adapted frame contain a basis $W_1(a), \dots, W_{n_s}(a)$ of every subspace $L^s(a)$ of the flag. The values of $W_1, \dots, W_n$ at a point b close to a also form a basis of $\mathbb{R}^n$. However, if a is singular, this basis may be not adapted to the flag $L^1(b) \subset L^2(b) \subset \cdots \subset L^{r(b)}(b) = \mathbb{R}^n$.

Définition 3.11 (*Privileged coordinates at a*)**.** A *system of privileged coordinates* at $a \in \Omega$ is a system of local coordinates $(z_1, \dots, z_n)$ centered at a (the image of a is 0) such that $\mathrm{ord}_a(z_j(x)) = w_j$, for $j = 1, \dots, n$.

Remark 3.5. For every system of local coordinates $(y_1, \dots, y_n)$ centered at a, we have, up to a re-ordering, $\mathrm{ord}_a(y_j) \leq w_j$ or, without re-ordering, $\sum_{j=1}^n \mathrm{ord}_a(y_j) \leq \sum_{j=1}^n w_j$.

The order at $a \in \Omega$ of functions and vector fields expressed in a system of privileged coordinates $(z_1, \dots, z_n)$ centered at a can be evaluated algebraically as follows :

- the order of the monomial $z_1^{\alpha_1} \dots z_n^{\alpha_n}$ is equal to its *weighted degree*

$$w(\alpha) := w_1\alpha_1 + \cdots + w_n\alpha_n;$$

- the order of a function $f(z)$ at $z = 0$ is the least weighted degree of the monomials occurring in the Taylor expansion of $f(z)$ at 0 ;
- the order of the monomial vector field $z_1^{\alpha_1} \dots z_n^{\alpha_n} \partial_{z_j}$ is equal to its *weighted degree* $w(\alpha) - w_j$, where one assigns the weight $-w_j$ to ∂_{z_j} at 0 ;
- the order of a vector field $W(z) = \sum_{j=1}^n W_j(z)\partial_{z_j}$ at $z = 0$ is the least weighted degree of the monomials occurring in the Taylor expansion of W at 0.

Définition 3.12 (*Continuously varying system of privileged coordinates*)**.** A *continuously varying system of privileged coordinates on* Ω is a mapping Φ taking values in $\mathbb{R}^n$, defined and continuous on a neighborhood of the set $\{(x, x), x \in \Omega\} \subset \Omega \times \Omega$, and so that the partial mapping $z := \Phi(a, \cdot)$ is a system of privileged coordinates at a. In this case, there exists a continuous function $\bar{\rho} : \Omega \to (0, +\infty)$ such that the coordinates $\Phi(a, \cdot)$ are defined on $B(a, \bar{\rho}(a))$. We call $\bar{\rho}$ an *injectivity radius function* of Φ.

Définition 3.13 (*Pseudo-norm*)**.** Let $a \in \Omega$ and $w_1, \dots, w_n$ the weights at a. The application from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\|z\|_a := |z_1|^{1/w_1} + \cdots + |z_n|^{1/w_n}, \qquad z = (z_1, \dots, z_n) \in \mathbb{R}^n,$$

is called *the pseudo-norm* at a.

3.2.1.3 Distance and error estimates

Privileged coordinates provide estimates of the sub-Riemannian distance d, according to the following result.

Theorem 3.3 (Ball-Box Theorem [11]). *Consider $(X_1, \dots, X_m) \in VF(\Omega)^m$, a point $a \in \Omega$, and a system of privileged coordinates z at a. There exist positive constants $C_d(a)$ and $\varepsilon_d(a)$ such that, for every $x \in \Omega$ with $d(a,x) < \varepsilon_d(a)$, one has*

$$\frac{1}{C_d(a)} \|z(x)\|_a \leq d(a,x) \leq C_d(a) \|z(x)\|_a. \tag{3.7}$$

If Ω contains only regular points and if Φ is a continuously varying system of privileged coordinates on Ω, then there exist continuous positive functions $C_d(\cdot)$ and $\varepsilon_d(\cdot)$ on Ω such that Eq. (3.7) holds true with $z = \Phi(a, \cdot)$ at all (x,a) satisfying $d(x,a) < \varepsilon_d(a)$.

Corollary 3.4. *Let K be a compact subset of Ω. Assume that K only contains regular points and there exists a continuously varying system of privileged coordinates Φ on K. Then, there exist positive constants C_K and ε_K such that, for every pair $(a,x) \in K \times K$ verifying $d(a,x) < \varepsilon_K$, one has*

$$\frac{1}{C_K} \|\Phi(a,x)\|_a \leq d(a,x) \leq C_K \|\Phi(a,x)\|_a. \tag{3.8}$$

Privileged coordinates also allow one to measure the error obtained when X is replaced by an approximation $\widehat{X}$.

Proposition 3.5 ([11, Prop. 7.29]). *Consider a point $a \in \Omega$, a system of privileged coordinates z at a, and an approximation $\widehat{X}$ of X at a. Then, there exist positive constants $C_e(a)$ and $\varepsilon_e(a)$ such that, for every $x \in \Omega$ with $d(a,x) < \varepsilon_e(a)$ and every integrable input function $u(\cdot)$ with $\ell(u) < \varepsilon_e(a)$, one has*

$$\|z(x(T,x,u)) - z(\hat{x}(T,x,u))\|_a \leq C_e(a) \max\left(\|z(x)\|_a, \ell(u)\right) \ell(u)^{1/r}, \tag{3.9}$$

where r is the degree of nonholonomy at a, $x(\cdot,x,u)$ and $\hat{x}(\cdot,x,u)$ are respectively the trajectories of $\dot{x} = \sum_{i=1}^m u_i X_i(x)$, and $\dot{x} = \sum_{i=1}^m u_i \widehat{X}_i(x)$.

3.2.2 Approximate steering method

Définition 3.14 (*Nonholonomic first order approximation on Ω*). A *nonholonomic first order approximation of X on Ω* is a mapping $\mathcal{A}$ which associates, with every $a \in \Omega$, a nonholonomic first order approximation of X at a defined on $B(a, \rho_a)$, i.e., $\mathcal{A}(a) := \widehat{X}^a$ on $B(a, \rho_a)$. The *approximation radius function* of $\mathcal{A}$ is the function $\rho : \Omega \to (0, \infty)$ which associates, with every a, its approximate radius ρ_a, i.e., $\rho(a) := \rho_a$.

In the sequel, *nonholonomic first-order approximations* will simply be called *approximations*. Useful properties of approximations are *continuity* and *nilpotency*.

Définition 3.15 (*Continuity and nilpotency of an approximation*). Let $\mathcal{A} : a \mapsto \widehat{X}^a$ be an approximation on Ω.

- We say that $\mathcal{A}$ is *continuous* if
  - (i) the mapping $(a,x) \mapsto \mathcal{A}(a)(x)$ is well-defined and, for every $a \in \Omega$, is continuous on a neighborhood of $(a,a) \in \Omega \times \Omega$;
  - (ii) the approximation radius function ρ of A is continuous.
- We say that $\mathcal{A}$ is *nilpotent of step $s \in \mathbb{N}$* if, for every $a \in \Omega$, the Lie algebra generated by $\widehat{X}^a$ is nilpotent of step s, i.e. every Lie bracket of length larger than s is equal to zero. (For a definition of the length of a Lie bracket, see Definitions 3.20 and 3.23.)

Consider a m-tuple of vector fields $X = \{X_1, \dots, X_m\}$ in $VF^m(\Omega)$.

Définition 3.16 (*Steering law of an approximation*)**.** Let

$$\mathcal{A} : a \mapsto \widehat{X}^a,$$

be an approximation of X on Ω and ρ its approximation radius function. A *steering law* of $\mathcal{A}$ is a mapping which, to every pair $(x, a) \in \Omega \times \Omega$ verifying $d(x, a) < \rho(a)$, associates an integrable input function $\hat{u} : [0, t] \mapsto \mathbb{R}^m$, henceforth called a *steering control*, such that the trajectory $\hat{x}(\cdot, x, \hat{u})$ of the *approximate control system*

$$\dot{x} = \sum_{i=1}^{m} u_i \widehat{X}_i^a(x), \tag{3.10}$$

is defined on $[0, T]$ and satisfies $\hat{x}(T, x, \hat{u}) = a$. In other words, $\hat{u}(\cdot)$ steers (3.10) from x to a.

A steering law of an approximation is intended to be used as an approximate steering law for the original system. For that purpose, it is important to have a continuity property of the steering control : the closest are x and a, the smaller is the length of $\hat{u}$. We introduce the stronger notion of *sub-optimality* (which is a sort of Lipschitz continuity of the steering law).

Définition 3.17 (*Sub-optimal steering law*)**.** Let $\mathcal{A}$ be an approximation of X on Ω and, for every $a \in \Omega$, let $\hat{d}_a$ be the sub-Riemannian distance associated to $\mathcal{A}(a)$. We say that a steering law of $\mathcal{A}$ is *sub-optimal* if there exists a constant $C_\ell > 0$ and a continuous positive function $\varepsilon_\ell(\cdot)$ such that, for any $a, x \in \Omega$ with $d(a, x) < \varepsilon_\ell(a)$, the control $\hat{u}(\cdot)$ steering (3.10) from x to a satisfies :

$$\ell(\hat{u}) \leq C_\ell \, \hat{d}_a(x, a) = C_\ell \, \hat{d}_a(\hat{x}(0, x, \hat{u}), \hat{x}(T, x, \hat{u})).$$

Note that, due to the definition of the sub-Riemannian distance $\hat{d}_a$, sub-optimal steering laws always exist.

Given an approximation $\mathcal{A}$ of X and a steering law for $\mathcal{A}$, we define a *local approximate steering* method for X as follows.

Définition 3.18 (*Local approximate steering*)**.** The *local approximate steering* (LAS for short) method associated to $\mathcal{A}$ and its steering law is the mapping $\mathrm{AppSteer}(\cdot, \cdot)$ which associates, with every pair $(x, a) \in \Omega \times \Omega$ verifying $d(x, a) < \rho(a)$, the point $x(T, x, \hat{u})$, i.e.,

$$\mathrm{AppSteer}(x, a) := x(T, x, \hat{u}),$$

where $\hat{u}(\cdot)$ is the steering control of $\mathcal{A}(a)$ associated to (x, a) and ρ is the approximation radius function of $\mathcal{A}$.

Définition 3.19 (*Local contractions and uniform local contractions*)**.** A LAS method is *locally contractive* if, for every $a \in \Omega$, there exist $\varepsilon_a > 0$ and $c_a < 1$ such that the following implication holds true :

$$d(a, x) < \varepsilon_a \implies d(a, \mathrm{AppSteer}(x, a)) \leq c_a d(a, x).$$

A LAS method is *uniformly locally contractive* on a compact set $K \subset \Omega$ if it is locally contractive, and if ε_a and c_a are independent of a, i.e., there exists $\varepsilon_K > 0$ and $c_K < 1$ such that, for every pair $(a, x) \in K \times K$, the following implication holds true :

$$d(a, x) < \varepsilon_K \implies d(a, \mathrm{AppSteer}(x, a)) \leq c_K d(a, x).$$

Remark 3.6. We will show that if $\widehat{X}$ is an approximation of X at a, the corresponding AppSteer function is locally contractive in a neighborhood of a. By the Fixed Point Theorem, one gets local convergence of Algorithm 1 (LAS). However, in order to obtain a globally convergent algorithm from LAS, one needs AppSteer to be *uniformly* locally contractive. In other words, the mapping $\mathcal{A}$ needs to be *continuous* in the sense of Definition 3.15.

As a direct consequence of Proposition 3.5, we obtain sufficient conditions for a LAS method to be uniformly locally contractive.

Corollary 3.6. *Let K be a compact subset of Ω. Assume that :*
(i) all points in K are regular ;
(ii) there exists a continuously varying system of privileged coordinates Φ on K ;
(iii) there exists a continuous approximation $\mathcal{A}$ of X on K ;
(iv) $\mathcal{A}$ is provided with a sub-optimal steering law.
Then, the LAS method AppSteer *associated to $\mathcal{A}$ and its steering law is uniformly locally contractive.*

Moreover, up to reducing the positive constant ε_K occurring in Corollary 3.4, one has, for every pair $(a, x) \in K \times K$ verifying $d(a, x) < \varepsilon_K$,

$$d(\text{AppSteer}(x, a), a) \leq \frac{1}{2} d(x, a), \tag{3.11}$$

$$\|z(\text{AppSteer}(x, a))\|_a \leq \frac{1}{2} \|z(x)\|_a. \tag{3.12}$$

Proof of Corollary 3.6. Under the hypotheses $(i) - (iv)$, one immediately extends Proposition 3.5 and obtains that there exist continuous positive functions $C_e(\cdot)$ and $\varepsilon_e(\cdot)$ such that inequality (3.9) holds true, with $z = \Phi(a, \cdot)$ and $\widehat{X} = \mathcal{A}(a)$, for every pair $(x, a) \in \Omega \times \Omega$ with $d(x, a) < \varepsilon_e(a)$ and every integrable input function $u(\cdot)$ with $\ell(u) < \varepsilon_e(a)$. The remaining argument is standard and one conclude easily.

□

Remark 3.7. Since the growth vector and the weights do not remain constant in any open neighborhood of a singular point, privileged coordinates z cannot vary continuously in any open neighborhood of that singular point. Therefore, around a singular point, the distance estimations provided in Eqs. (3.8) and (3.12) and based on privileged coordinates do not hold true uniformly. In particular, if (a_n) is a sequence of regular points converging to a singular point a (this is possible since regular points are dense in Ω), the sequences $\varepsilon_d(a_n)$ and $\varepsilon_e(a_n)$ tend to zero whereas $\varepsilon_d(a)$ and $\varepsilon_e(a)$ are not equal to zero.

Remark 3.8. A similar discontinuity issue occurs of course for the approximate system. Indeed, if a is a singular point, the growth vector and the weights of the associated privileged coordinates at a change around a, implying a change of the truncation order in the Taylor expansion of the vector fields. Therefore, the approximate vector fields cannot vary continuously in any neighborhood of a singular point.

3.3 Desingularization by Lifting

As it appears in Corollary 3.6, the absence of singular points is one of the key features in order to construct uniformly locally contractive LAS method. As a matter of fact, we will show in Section 3.4.2 how to construct a globally convergent motion planning algorithm for a regular nonholonomic system (i.e., when all points in Ω are regular).

However, in general, nonholonomic systems do have singular points. For such systems, attempts have been made to construct specific LAS methods (see [106, 53]), but additional conditions on the structure of the singularities are required. Our approach here is different : we present in this section a desingularization procedure of the system, in such a way to replace a MPP for a non regular system by a MPP for a regular one.

The strategy consists in "*lifting*" the vector fields

$$\{X_1, \dots, X_m\} \in VF^m(\Omega)$$

defining the control system to some extended domain $\widetilde{\Omega} := \Omega \times \mathbb{R}^{\tilde{n}}$, with $\tilde{n} \in \mathbb{N}$ to be defined later. The lifted vector fields

$$\{\xi_1, \dots, \xi_m\} \in VF^m(\widetilde{\Omega})$$

are constructed so that :

(i) for $i = 1, \dots, m$, ξ_i has the following form,

$$\xi_i(x, y) = X_i(x) + \sum_{j=1}^{\tilde{n}} b_{ij}(x, y)\partial_{y_j}, \qquad (x, y) \in \Omega \times \mathbb{R}^{\tilde{n}},$$

where the functions b_{ij}, $j = 1, \dots, \tilde{n}$, are smooth ;

(ii) the Lie algebra generated by $\{\xi_1, \dots, \xi_m\}$ is free up to step r (see Def. 3.25 below).

Point (ii) guarantees that the nonholonomic system defined by

$$\{\xi_1, \dots, \xi_m\}$$

is regular, since its growth vector is constant on $\widetilde{\Omega}$. Point (i) guarantees that one obtains $X_1, \dots, X_m$ by *projecting* $\xi_1, \dots, \xi_m$ on $\mathbb{R}^n$. Indeed, let π be the canonical projector from $\widetilde{\Omega}$ to Ω defined by $\pi(\widetilde{x}) = x$, where $\widetilde{x} = (x, y) \in \widetilde{\Omega}$. Then, denoting $d\pi_{\widetilde{x}}$ the differential of π at $\widetilde{x}$, one has

$$d\pi_{\widetilde{x}}(\xi_i(\widetilde{x})) = X_i(\pi(\widetilde{x})).$$

As a consequence, the projection by π of a trajectory $\widetilde{x}(\cdot, \widetilde{x}_0, u)$ of the control system

$$\dot{\widetilde{x}} = \sum_{i=1}^{m} u_i \xi_i(\widetilde{x}), \widetilde{x} \in \widetilde{\Omega}, \tag{3.13}$$

is a trajectory of (4.5) associated to the *same* input, i.e.,

$$\pi\big(\widetilde{x}(\cdot, \widetilde{x}_0, u)\big) = x(\cdot, \pi(\widetilde{x}_0), u).$$

Therefore, any control u steering System (3.13) from a point $\widetilde{x}_0 := (x_0, 0)$ to a point $\widetilde{x}_1 := (x_1, 0)$ also steers System (4.5) from x_0 to x_1. It then suffices to solve the MPP for the regular System (3.13).

Note that distinguished desingularization procedures already exist, cf. [84, 11, 52]. However, an important property of the desingularization procedure presented here is that all the changes of coordinates and intermediate constructions involved in it are explicit and purely algebraic. Note also that, during the lifting process, we obtain, as a byproduct, a nonholonomic first order approximation of $\{\xi_1, \dots, \xi_m\}$ in a "canonical" form, which can be exactly controlled by sinusoids (see Chapter 3.5).

We start this section by presenting some general facts on free Lie algebras, namely the *P. Hall basis* in Section 3.3.1, and the *canonical form* of a nilpotent Lie algebra of step r in Section 3.3.2. We then give one *desingularization procedure* in Section 3.3.3. The proofs of the results stated in Section 3.3.3 will be gathered in Section 3.3.4.

3.3.1 P. Hall basis on a free Lie algebra and evaluation map

In this section, we present some general facts on free Lie algebras. The reader is referred to [19] for more details. Consider $\mathcal{I} := \{1, \dots, m\}$, and the free Lie algebra $\mathcal{L}(\mathcal{I})$ generated by the elements of $\mathcal{I}$. Recall that $\mathcal{L}(\mathcal{I})$ is the $\mathbb{R}$-vector space generated by the elements of $\mathcal{I}$ and their formal brackets $[\ ,\ ]$, together with the relations of skew-symmetry and the Jacobi identity enforced (see [19] for more details). We note that, by construction, for every $I \in \mathcal{L}(\mathcal{I})$, there exists $(I_1, I_2) \in \mathcal{L}(\mathcal{I}) \times \mathcal{L}(\mathcal{I})$ such that $I = [I_1, I_2]$.

Définition 3.20 (Length of the elements of $\mathcal{L}(\mathcal{I})$)**.** The *length* of an element I of a free Lie algebra $\mathcal{L}(\mathcal{I})$, denoted by $|I|$, is defined inductively by

$$|I| := 1, \text{ for } I = 1, \dots, m; \tag{3.14}$$

$$|I| := |I_1| + |I_2|, \text{ for } I = [I_1, I_2], \text{ with } I_1, I_2 \in L(X). \tag{3.15}$$

We use $\mathcal{L}^s(\mathcal{I})$ to denote the subspace generated by elements of $\mathcal{L}(\mathcal{I})$ of length not greater than s. Let $\widetilde{n}_s$ be the dimension of $\mathcal{L}^s(\mathcal{I})$.

The *P. Hall basis* of $\mathcal{L}(\mathcal{I})$ is a totally ordered subset of $\mathcal{L}(\mathcal{I})$ defined as follows.

Définition 3.21 (*P. Hall basis*)**.** A subset $\mathcal{H} = \{I_j\}_{j \in \mathbb{N}}$ of $\mathcal{L}(\mathcal{I})$ is the *P. Hall basis* of $\mathcal{L}(\mathcal{I})$ if (H1), (H2), (H3), and (H4) are verified.

(H1) If $|I_i| < |I_j|$, then $I_i \prec I_j$;

(H2) $\{1, \dots, m\} \subset \mathcal{H}$, and we impose that $1 \prec 2 \prec \cdots \prec m$;

(H3) every element of length 2 in $\mathcal{H}$ is in the form $[I_i, I_j]$ with $(I_i, I_j) \in \mathcal{I} \times \mathcal{I}$ and $I_i \prec I_j$;

(H4) an element $I_k \in \mathcal{L}(\mathcal{I})$ of length greater than 3 belongs to $\mathcal{H}$ if $I_k = [I_{k_1}, [I_{k_2}, I_{k_3}]]$ with $I_{k_1}, I_{k_2}, I_{k_3}$, and $[I_{k_2}, I_{k_3}]$ belonging to $\mathcal{H}$, $I_{k_2} \prec I_{k_3}$, $I_{k_2} \prec I_{k_1}$ or $I_{k_2} = I_{k_1}$, and $I_{k_1} \prec [I_{k_2}, I_{k_3}]$.

The elements of $\mathcal{H}$ form a basis of $\mathcal{L}(\mathcal{I})$, and " $\prec$ " defines a strict and total order over the set $\mathcal{H}$. In the sequel, we use I_k to denote the k^{th} element of $\mathcal{H}$ with respect to the order " $\prec$ ". Let $\mathcal{H}^s$ be the subset of $\mathcal{H}$ of all the elements of length not greater than s. The elements of $\mathcal{H}^s$ form a basis of $\mathcal{L}^s(\mathcal{I})$ and $\text{Card}(\mathcal{H}^s) = \widetilde{n}_s$. The set $\mathcal{G}^s := \mathcal{H}^s \setminus \mathcal{H}^{s-1}$ contains the elements in $\mathcal{H}$ of length equal to s. Its cardinal will be denoted by $\widetilde{k}_s = \text{Card}(\mathcal{G}^s)$.

By (H1)−(H4), every element $I_j \in \mathcal{H}$ can be expanded in a unique way as

$$I_j = [I_{k_1}, [I_{k_2}, \cdots, [I_{k_i}, I_k] \cdots]], \tag{3.16}$$

with $k_1 \geq \cdots \geq k_i$, $k_i < k$, and $k \in \{1, \dots, \tilde{n}_1\}$. In that case, the element I_j is said to be a *direct descendent* of I_k, and we write $\phi(j) := k$. For $I_j \in \mathcal{H}^r$, the expansion (3.16) also associates with $I_j \in \mathcal{H}$ a sequence $\alpha_j = (\alpha_j^1, \dots, \alpha_j^{\widetilde{n}_r})$ in $\mathbb{Z}^{\widetilde{n}_r}$ defined by

$$\alpha_j^\ell := \text{ Card } \{s \in \{1, \dots, i\}, k_s = \ell\}.$$

By construction, one has $\alpha_j^\ell = 0$ for $\ell \geq j$, and $\alpha_j = (0, \dots, 0)$ for $1 \leq j \leq \tilde{n}_1$.

Consider now a family of m vector fields $X = \{X_1, \dots, X_m\}$ and the Lie algebra $L(X)$ they generate. The P. Hall basis $\mathcal{H}$ induces, via the *evaluation map*, a family of vector fields spanning $L(X)$ as a linear space.

Définition 3.22 (*Evaluation map*)**.** The *evaluation map* E_X defined on $\mathcal{L}(\mathcal{I})$, with values in $L(X)$, assigns to every $I \in \mathcal{L}(\mathcal{I})$ the vector field $X_I = E_X(I)$ obtained by plugging in X_i, $i = 1, \dots, m$, for the corresponding letter i.

Définition 3.23 (*Length of the elements of $L(X)$*)**.** With the notations of Definition 3.22, if $X_I = E_X(I)$, the length of X_I, denoted by $\Delta(X_I)$, is set to be equal to $|I|$, .

Définition 3.24 (*P. Hall family*)**.** The *P. Hall family* H_X associated with the vector fields $X = \{X_1, \ldots, X_m\}$ is defined by

$$H_X := \{E_X(I), I \in \mathcal{H}\},$$

where E_X is the evaluation map and $\mathcal{H}$ is the P. Hall basis of the free Lie algebra $\mathcal{L}(\mathcal{I})$. The family H_X inherits the ordering and the numbering of the elements in $\mathcal{H}$ induced by (H1)–(H4).

Note that H_X is a spanning set of $L(X)$, but not always a basis.

Définition 3.25 (*Free up to step s*)**.** Let s be a positive integer such that $1 \leq s \leq r$. A family of vector fields $\xi = \{\xi_1, \ldots, \xi_m\}$ defined on a subset $\widetilde{\Omega}$ of $\mathbb{R}^{\widetilde{n}_r}$ is said to be *free up to step s* if, for every $\widetilde{x} \in \widetilde{\Omega}$, the growth vector $(n_1(\widetilde{x}), \ldots, n_s(\widetilde{x}))$ is equal to $(\widetilde{n}_1, \ldots, \widetilde{n}_s)$.

Remark 3.9. If ξ defined on $\widetilde{\Omega} \subset \mathbb{R}^{\widetilde{n}_r}$ is free up to step r, then every point of $\widetilde{\Omega}$ is *regular*.

Définition 3.26 (*Free weights*)**.** Let $\xi = \{\xi_1, \ldots, \xi_m\}$ be free up to step r on $\widetilde{\Omega} \subset \mathbb{R}^{\widetilde{n}_r}$. The integers $\widetilde{w}_1, \ldots, \widetilde{w}_{\widetilde{n}_r}$, where $\widetilde{w}_j = s$ if $n_{s-1}(\widetilde{x}) < j \leq n_s(\widetilde{x})$ for every $\widetilde{x} \in \widetilde{\Omega}$, are called the *free weights* of step r.

3.3.2 Canonical form

We present in this section the construction of a canonical form for nilpotent systems proposed by Grayson and Grossman in [40] and [41]. Similar results were also obtained by Sussmann in [94].

The construction takes place in $\mathbb{R}^{\widetilde{n}_r}$, where r is a positive integer and $\widetilde{n}_r$ the dimension of $\mathcal{L}^r(\mathcal{I})$. We denote by $v = (v_1, \ldots, v_{\widetilde{n}_r})$ the points in $\mathbb{R}^{\widetilde{n}_r}$, and by $(\partial_{v_1}, \ldots, \partial_{v_{\widetilde{n}_r}})$ the canonical basis of $\mathbb{R}^{\widetilde{n}_r}$. For $j = 1, \ldots, \widetilde{n}_r$, we assign to the coordinate function v_j the weight $\widetilde{w}_j$ and to the vector ∂_{v_j} the weight $-\widetilde{w}_j$. The *weighted degree* of a monomial $v_1^{\alpha_1} \cdots v_{\widetilde{n}_r}^{\alpha_{\widetilde{n}_r}}$ is then defined as

$$\widetilde{w}(\alpha) := \widetilde{w}_1\alpha_1 + \cdots + \widetilde{w}_{\widetilde{n}_r}\alpha_{\widetilde{n}_r}, \qquad \text{where } \alpha = (\alpha_1, \ldots, \alpha_{\widetilde{n}_r}),$$

and the weighted degree of a monomial vector field $v_1^{\alpha_1} \ldots v_{\widetilde{n}_r}^{\alpha_{\widetilde{n}_r}} \partial_{v_j}$ is defined as $\widetilde{w}(\alpha) - \widetilde{w}_j$.

For every $I_j \in \mathcal{H}^r$, let α_j be the sequence associated with I_j (see Subsection 3.3.1). Define the monomial $P_j(v)$ associated with I_j by

$$P_j(v) := \frac{v^{\alpha_j}}{\alpha_j!}, \tag{3.17}$$

where $v^{\alpha_j} := \prod_\ell v_\ell^{\alpha_j^\ell}$, and $\alpha_j! := \prod_\ell \alpha_j^\ell!$. Note that P_j satisfies the following inductive formulas.

$$\begin{array}{rcll} P_j(v) & = & 1 & \text{if } I_j \in \mathcal{H}^1; \\ P_j(v) & = & \dfrac{v_{j_1}}{\alpha_{j_2}^{j_1} + 1} P_{j_2}(v) & \text{if } I_j = [I_{j_1}, I_{j_2}]. \end{array} \tag{3.18}$$

Theorem 3.7 ([40, 41]). *We define the vector fields $D_1, \dots, D_m$ on $\mathbb{R}^{\widetilde{n}_r}$ as follows :*

$$\begin{aligned} D_1 &= \partial_{v_1}, \\ D_2 &= \partial_{v_2} + \sum_{\substack{2\le|I_j|\le r \\ \phi(j)=2}} P_j(v)\partial_{v_j}, \\ &\vdots \\ D_m &= \partial_{v_m} + \sum_{\substack{2\le|I_j|\le r \\ \phi(j)=m}} P_j(v)\partial_{v_j}. \end{aligned}$$

Then, the Lie algebra $L(D)$ generated by $(D_1, \dots, D_m)$ is free up to step r, and one has

$$D_{I_j}(0) = \partial_{v_j}, \quad \text{for } I_j \in \mathcal{H}^r,$$

where $D_{I_j} := E_D(I_j)$ is defined through the evaluation map E_D with values in $L(D)$.

The proof of Theorem 3.7 goes by induction on the length of elements in $L(D)$. The reader is referred to [41] for a complete development.

Corollary 3.8. *For all $I_k \in \mathcal{H}^r$, D_{I_k} has the following form*

$$D_{I_k} = \partial_{v_k} + \sum_{I_j\in\mathcal{H}^r,\ |I_j|>|I_k|} P_j^k(v)\partial_{v_j}, \tag{3.19}$$

where every non zero polynomial P_j^k is homogeneous of weighted degree equal to $|I_j| - |I_k|$.

Remark 3.10. The explicit expression of the polynomials $P_j^k(v)$ as functions of the monomials $P_j(v)$ is obtained through an induction formula.

Corollary 3.9. *For $i = 1, \dots, m$, we define m vector fields $\check{D}_i$ as follows :*

$$\check{D}_i := \partial_{v_i} + \sum_{\substack{2\le|I_k|\le\mathcal{H}^{r-1} \\ \phi(k)=i}} P_k(v)\partial_{v_k} + \sum_{\substack{I_j\in S \\ \phi(j)=i}} P_j(v)\partial_{v_j},$$

where S is an arbitrary non-empty subset of $\mathcal{G}^r$. Then,

- *if $I_k \in \mathcal{H}^{r-1} \cup S$, we have*

$$\check{D}_{I_k} = \partial_{v_k} + \sum_{I_j\in\mathcal{H}^{r-1}\cup S,\ |I_j|>|I_k|} P_j^k(v)\partial_{v_j};$$

- *if $I_k \in \mathcal{G}^r \setminus S$, we have $\check{D}_{I_k} = 0$.*

Définition 3.27 (*Canonical form*). Let $X_1, \dots, X_m$ be m vector fields on an open subset Ω of $\mathbb{R}^{\widetilde{n}_r}$ and v a local system of coordinates on Ω. The control system associated to $\{X_1, \dots, X_m\}$ is said to be in *canonical form in the coordinates v* if one has

$$v_*X_i = D_i, \quad \text{for } i = 1, \dots, m,$$

where we use v_*X_i to denote the push-forward of X_i by v.

Consider now the control system given by

$$\dot{v} = \sum_{i=1}^{m} u_i D_i(v), \quad v \in \mathbb{R}^{\widetilde{n}_r}. \tag{3.20}$$

Writing (3.20) component by component, one has, for $j = 1, \ldots, \widetilde{n}_r$,

$$\dot{v}_j = P_j(v_1, \ldots, v_{j-1})u_i, \quad \text{where } i = \phi(j), \tag{3.21}$$

or inductively,

$$\dot{v}_j = \frac{v_{j_1}}{\alpha^{j_1}_{j_2} + 1}\dot{v}_{j_2}, \quad \text{where } I_j = [I_{j_1}, I_{j_2}]. \tag{3.22}$$

More explicitly, one has

$$\dot{v}_j = \frac{1}{k!}v^k_{j_1}\dot{v}_{j_2}, \quad \text{if } X_{I_j} = \mathrm{ad}^k_{X_{I_{j_1}}} X_{I_{j_2}}, \tag{3.23}$$

where $\mathrm{ad}^k_{X_{I_{j_1}}} X_{I_{j_2}} := [\underbrace{X_{I_{j_1}}, [X_{I_{j_1}}, \cdots, [X_{I_{j_1}}}_{k \text{ times}}, X_{I_{j_2}}]$,

with $X_{I_{j_2}} = [X_{I_{j_3}}, X_{I_{j_4}}]$ and $I_{j_3} \prec I_{j_1}$. The inductive formula (3.23) will be used in Chapter 3.5.

A particular system of coordinates, called *canonical coordinates* (a terminology arising from Lie group theory), allows one to obtain canonical forms. Consider m vector fields $X_1, \ldots, X_m$ on $\Omega \subset \mathbb{R}^{\widetilde{n}_r}$, let $v \in \Omega$, and $W = \{W_1, \ldots, W_n\}$ be a set of vector fields in $L(X)$ such that $W_1(v), \ldots, W_n(v)$ is a basis of $\mathbb{R}^{\widetilde{n}_r}$. The *canonical coordinates of the second kind* at v associated with W are the system of local coordinates at v defined as the inverse of the local diffeomorphism

$$(z_1, \ldots, z_{\widetilde{n}_r}) \longmapsto e^{z_{\widetilde{n}_r} W_{\widetilde{n}_r}} \circ \cdots \circ e^{z_1 W_1}(v), \tag{3.24}$$

where we use e^{zW_i} to denote the flow of W_i. When the system

$$(X_1, \ldots, X_m)$$

is nilpotent, the above diffeomorphism defines global coordinates on Ω for every $v \in \Omega$.

Theorem 3.10 ([94]). *Assume that the vector fields $(X_1, \ldots, X_m)$ generate a Lie algebra which is both nilpotent of step r and free up to step r. Then, in the canonical coordinates of the second kind associated with the P. Hall basis H^r_X, the control system defined by $(X_1, \ldots, X_m)$ is in canonical form.*

Remark 3.11. The canonical coordinates of the second kind require to determine the flow of the control vector fields i.e., to integrate some differential equations. In general, there does not exist algebraic change of coordinates between an arbitrary system of coordinates and the canonical coordinates of the second kind.

3.3.3 Desingularization algorithm

Let $X = \{X_1, \ldots, X_m\} \subset VF(\Omega)$ be a family of m vector fields on $\Omega \subset \mathbb{R}^n$, and K be a compact subset of Ω. We assume that the LARC is satisfied at every point of K. Therefore, the degree of nonholonomy of X is *bounded* on K and we denote by r its maximal value.

Recall that $\mathcal{H}^r$ denote the elements of the P. Hall basis of length smaller or equal to r. For every n-tuple $\mathcal{J} = (I_1, \ldots, I_n)$ of elements of $\mathcal{H}^r$, we define the domain $\mathcal{V}_\mathcal{J} \subset \Omega$ by

$$\mathcal{V}_\mathcal{J} := \{\ p \in \Omega \text{ such that } \det(X_{I_1}(p), \ldots, X_{I_n}(p)) \neq 0\ \}, \tag{3.25}$$

where $X_{I_j} = E_X(I_j)$. Such a set $\mathcal{V}_\mathcal{J}$ is open in Ω (possibly empty) and for every $p \in \mathcal{V}_\mathcal{J}$, the vectors $X_{I_1}(p), \ldots, X_{I_n}(p)$ form a basis of $\mathbb{R}^n$.

Since K is compact, there exist a finite family of n-tuples $\mathcal{J}_1, \ldots, \mathcal{J}_M$ of elements of $\mathcal{H}^r$ such that

$$K \subset \bigcup_{i=1}^{M} \mathcal{V}_{\mathcal{J}_i}. \tag{3.26}$$

One deduces from (3.26) a compact covering of K in the form

$$K \subset \bigcup_{i=1}^{M} \mathcal{V}^c_{\mathcal{J}_i}, \tag{3.27}$$

where, for $i = 1, \ldots, M$, the set $\mathcal{V}^c_{\mathcal{J}_i} \subset \mathcal{V}_{\mathcal{J}_i}$ is compact.

Définition 3.28. Let $(S_i)_{i\in I}$ be a finite set of subsets of Ω. The *connectedness graph* $\mathsf{G} := (\mathsf{N}, \mathsf{E})$ associated with $(S_i)_{i\in I}$ is defined as as follows :

- the set of nodes $\mathsf{N} := I$;
- a pair (i, j) with i and j in N belongs to the set of edges E if $S_i \cap S_j \neq \emptyset$.

A *simple path* on G is a subset $\mathsf{p} := \{i_1, \ldots, i_L\}$ of two by two distinct elements of N such that, for $j = 1, \ldots, L-1$, the pair (i_j, i_{j+1}) belongs to E.

Remark 3.12. With the notations of Definition 3.28, if we assume that all the sets S_i are open or, all of them are closed, and the set $S := \cup_{i\in I} S_i$ is connected, then, for every $(x_0, x_1) \in S \times S$, there exists a simple path on G denoted by $p := \{i_1, \ldots, i_L\}$ such that $x_0 \in S_{i_1}$ and $x_1 \in S_{i_L}$.

Take $\mathcal{J} = (I_1, \ldots, I_n)$ among $\mathcal{J}_1, \ldots, \mathcal{J}_M$, and pick a point a in $\mathcal{V}_\mathcal{J}$. In the sequel, we construct, by induction on the length of elements in a free Lie algebra, a family of m vector fields $\xi = \{\xi_1, \ldots, \xi_m\}$ defined on $\mathcal{V}_\mathcal{J} \times \mathbb{R}^{\tilde{n}_r - n}$, which is free up to step r and has its projection on $\mathcal{V}_\mathcal{J}$ equal to X. At the same time, we give an approximation of ξ at $\tilde{a} := (a, 0) \in \mathcal{V}_\mathcal{J} \times \mathbb{R}^{\tilde{n}_r - n}$ in canonical form.

We define

$$\begin{aligned} \mathcal{J}^s &:= \{I_j \in \mathcal{J}, \text{ with } |I_j| = s\}, \qquad \text{for } s \geq 1, \\ \mathcal{G}^s &:= \mathcal{H}^s \setminus \mathcal{H}^{s-1}, \qquad \text{for } s \geq 2. \end{aligned}$$

We denote by k_s the cardinal of $\mathcal{J}^s$, and by $\tilde{k}_s$ the cardinal of $\mathcal{G}^s$. We are now ready to describe in details our desingularization algorithm.

Desingularization Algorithm (DA)

– **Step 1 :**

(1-1) Set $\mathcal{V}^1 := \mathcal{V}_\mathcal{J} \times \mathbb{R}^{\tilde{k}_1 - k_1}$, $a^1 := (a, 0) \in \mathcal{V}^1$, $\mathcal{K}^1 := \mathcal{H}^1 \cup (\mathcal{J} \setminus \mathcal{J}^1)$.

(1-2) Define $\{\xi_1^1, \ldots, \xi_m^1\}$ on $\mathcal{V}^1$ as follows :

$$\forall (x, v^1) \in \mathcal{V}^1, \qquad \xi_i^1(x, v^1) := X_i(x) + \begin{cases} 0, & \text{for } i \in \mathcal{J}^1, \\ \partial_{v_i^1}, & \text{for } i \in \mathcal{G}^1 \setminus \mathcal{J}^1. \end{cases}$$

(1-3) Compute the coordinates y^1 on $\mathcal{V}^1$ defined as the unique affine system of coordinates on $\mathcal{V}^1$ such that

$$\partial_{y_j^1} = \xi_{I_j}^1(a^1) \quad \text{for } I_j \in \mathcal{K}^1, \quad \text{and} \quad y^1(a^1) = 0.$$

(1-4) Construct the system of global coordinates z^1 on $\mathcal{V}^1$ by

$$\begin{aligned} z_j^1 &:= y_j^1, \quad \text{for } j \in \mathcal{H}^1, \\ z_j^1 &:= y_j^1 - \sum_{k=1}^{\tilde{n}_1} (\xi_k^1 \cdot y_k^1)(y^1)_{|y^1=0} \ y_k^1, \quad \text{for } I_j \in \mathcal{K}^1 \setminus \mathcal{H}^1, \end{aligned}$$

where I_j denotes the j^{th} element in $\mathcal{K}^1$.

– **Step s, $2 \leq s \leq r$:**

(s-1) Set $\mathcal{V}^s := \mathcal{V}^{s-1} \times \mathbb{R}^{\tilde{k}_s - k_s}$, $a^s := (a, 0) \in \mathcal{V}^s$, and $\mathcal{K}^s := \mathcal{K}^{s-1} \cup (\mathcal{G}^s \setminus \mathcal{J}^s)$. Denote by v^s the points in $\mathbb{R}^{\tilde{k}_s - k_s}$.

(s-2) Define $\{\xi_1^s, \ldots, \xi_m^s\}$ as the vector fields on $\mathcal{V}^s$ which write in coordinates (z^{s-1}, v^s) as :

$$\xi_i^s(z^{s-1}, v^s) = \xi_i^{s-1}(z^{s-1}) + \sum_{\substack{I_k \in \mathcal{G}^s \setminus \mathcal{J}^s \\ \phi(k) = i}} P_k(z^{s-1}) \partial_{v_k^s} \ .$$

(s-3) Compute the system of global coordinates y^s on $\mathcal{V}^s$ as the unique isomorphism $(z^{s-1}, v^s) \mapsto y^s$ such that

$$\partial_{y^s_{\phi(I)}} = \xi_I^s(a^s) \quad \text{for every } I \in \mathcal{K}^s \ .$$

(s-4) Construct the system of global coordinates $\tilde{z}^s$ on $\mathcal{V}^s$ by the following recursive formulas :
(s-4)-(a) for $I_j \in \mathcal{H}^s$,

$$\tilde{z}_j^s := y_j^s + \sum_{k=2}^{|I_j|-1} r_k(y_1^s, \ldots, y_{j-1}^s), \tag{3.28}$$

where, for $k = 2, \ldots, |I_j| - 1$,

$$\begin{aligned} & r_k(y_1^s, \ldots, y_{j-1}^s) \\ = \ & - \sum_{\substack{|\beta| = k \\ \omega(\beta) < |I_j|}} \big[(\xi_{I_1}^s)^{\beta_1} \cdots (\xi_{I_{j-1}}^s)^{\beta_{j-1}} \cdot (y_j^s + \sum_{q=2}^{k-1} r_q) \big] (y^s)_{|y^s=0} \\ & \frac{(y_1^s)^{\beta_1}}{\beta_1!} \cdots \frac{(y_{j-1}^s)^{\beta_{j-1}}}{\beta_{j-1}!} \ ; \end{aligned}$$

(s-4)-(b) for $I_j \in \mathcal{K}^s \setminus \mathcal{H}^s$,

$$\tilde{z}^s_j := y^s_j + \sum_{k=2}^{s} r_k(y^s_1, \ldots, y^s_{\widetilde{n}_s}), \tag{3.29}$$

where, for $k = 2, \ldots, s$,

$$\begin{aligned} & r_k(y^s_1, \ldots, y^s_{\widetilde{n}_s}) \\ = & - \sum_{\substack{|\beta|=k \\ \omega(\beta)\leq s}} \big[(\xi^s_{I_1})^{\beta_1} \cdots (\xi^s_{I_{\widetilde{n}_s}})^{\beta_{\widetilde{n}_s}} \cdot (y^s_j + \sum_{q=2}^{s} r_q)\big](y^s)_{|y^s=0} \\ & \frac{(y^s_1)^{\beta_1}}{\beta_1!} \cdots \frac{(y^s_{j-1})^{\beta_{\widetilde{n}_s}}}{\beta_{\widetilde{n}_s}!}. \end{aligned}$$

(s-5) Construct the system of global coordinates z^s as follows :

(s-5)-(a) for $j > \widetilde{n}_s$, set $z^s_j := \tilde{z}^s_j$;

(s-5)-(b) for $j = 1, \ldots, \widetilde{n}_s$, set

$$z^s_j := \Psi^s_j(\tilde{z}^s_1, \ldots, \tilde{z}^s_{j-1}),$$

where all Ψ^s_j are polynomials such that the two following conditions are satisfied :

- if we impose the weight of z^s_j to be $\widetilde{w}_j$ for $j = 1, \ldots, \widetilde{n}_s$, then every Ψ^s_j is homogeneous of weighted degree equal to $\widetilde{w}_j$;
- denote by $\mathrm{ord}^s_{a^s}(\cdot)$ the nonholonomic order defined by $(\xi^s_1, \ldots, \xi^s_m)$ at a^s, and by $\xi^s_{i,j}(z^s)$ the j^{th} component of $\xi^s_i(z^s)$; then one has

$$\xi^s_{i,j}(z^s) = \delta_{i,\phi(j)} P_j(z^s_1, \ldots, z^s_{j-1}) + R_{i,j}(z^s), \quad j = 1, \ldots, \widetilde{n}_s, \tag{3.30}$$

where $\mathrm{ord}^s_{a^s}(R_{i,j}) \geq \widetilde{w}_j$ ($\delta_{i,k}$ denotes the Kronecker symbol). Note that $\mathrm{ord}^s_{a^s}(P_j) = \widetilde{w}_j - 1$.

Theorem 3.11. *Let $\xi_i := \xi^r_i$, for $i = 1, \ldots, m$, and $z := z^r$, where ξ^r_i and z^r are given by the desingularization algorithm. Then, the family of vector fields $\xi = \{\xi_1, \ldots, \xi_m\}$ defined on $\Omega \times \mathbb{R}^{\widetilde{n}_r - n}$ is free up to step r. Moreover, the system of coordinates $z = (z_1, \ldots, z_{\widetilde{n}_r})$ is a system of privileged coordinates at $\tilde{a}$ for ξ, and the family of vector fields $\widehat{\xi} = \{\widehat{\xi}_1, \ldots, \widehat{\xi}_m\}$ defined in the coordinates z by the canonical form :*

$$\widehat{\xi}_i = \partial_{z_i} + \sum_{\substack{2 \leq |I_j| \leq \widetilde{n}_r \\ \phi(j)=i}} P_j(z_1, \ldots, z_{j-1}) \partial_{z_j}, \quad \textit{for } i = 1, \ldots, m, \tag{3.31}$$

is a nonholonomic first order approximation of ξ at $\tilde{a}$.

Remark 3.13. We note that the desingularization procedure does not a priori require that

(a) the coordinates z are privileged coordinates ;

(b) the system $\widehat{\xi}$ is a first order approximation of ξ at a.

However, (a) and (b) can be used directly at the first step of the motion planning algorithm presented in Chapter 3.4.

Remark 3.14. If we assume that the original system $X = \{X_1, \dots, X_m\}$ is nilpotent, then, by adapting the proof of Theorem 3.11 presented in Section 3.3.4, one can show that the corresponding "lifted" system $\xi = \{\xi_1, \dots, \xi_m\}$ given by the Desingularization Algorithm proposed in this section remains *nilpotent* with the same order of nilpotency. Moreover, when expressed in the privileged coordinates z, the system ξ is equal to its own first order approximation in the canonical form. In other words, for any nilpotent systems of step k, the Desingularization Algorithm constructs a nilpotent system of step k and free up to step k which is in the canonical form in coordinates z.

3.3.4 Proof of Theorem 3.5

The proof of Theorem 3.11 is based on the following proposition.

Proposition 3.12. *The desingularization algorithm is feasible from $s = 1$ to $s = r$. At each step s of the construction $(s = 1, \dots, r)$, the following properties hold true :*

(A1) *the vectors $\{\xi^s_I(a^s) \ : \ I \in \mathcal{K}^s\}$ are linearly independent ;*

(A2) *if $|I_j| \leq s$, then $\mathrm{ord}^s_{a^s}(\tilde{z}^s_j) = |I_j|$, and $\mathrm{ord}^s_{a^s}(z^s_j) = |I_j|$;*

(A3) *if $|I_j| > s$, then $\mathrm{ord}^s_{a^s}(z^s_j) > s$;*

(A4) *the change of coordinates $(\Psi^s_j)_{j=1,\dots,\widetilde{n}_s}$ is well defined ;*

(A5) *for $I_k \in \mathcal{K}^s$, the vector fields $\xi^s_{I_k}$ has the following form in coordinates z^s,*

$$\xi^s_{I_k}(z^s) = \sum_{I_j \in \mathcal{H}^s} (P^k_j(z^s) + R^k_j(z^s))\partial_{z^s_j} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q^k_\ell(z^s)\partial_{z^s_\ell}, \tag{3.32}$$

with $\mathrm{ord}^s_{a^s}(R^k_j) > |I_j| - |I_k|$, $\mathrm{ord}^s_{a^s}(Q^k_\ell) > s - |I_k|$, and P^k_j given by Eq. (3.19).
More precisely, if one defines $\check{\xi}^s_i$ by

$$\check{\xi}^s_i := \sum_{\substack{I_j \in \mathcal{H}^s \\ \phi(j) = i}} P_j(z^s)\partial_{z^s_j},$$

then, one has

$$\check{\xi}^s_{I_k} = \sum_{I_j \in \mathcal{H}^s} P^k_j(z^s)\partial_{z^s_j},$$

where the polynomials P^k_j verify the following properties :
- *if $I_k \in \mathcal{H}^s$, then*
  - *for $|I_j| < |I_k|$, $P^k_j = 0$;*
  - *for $|I_j| = |I_k|$, $P^j_j = 1$, and $P^k_j = 0$ if $k \neq j$;*
  - *for $|I_j| > |I_k|$, $\mathrm{ord}^s_{a^s}(P^k_j) = |I_j| - |I_k|$;*
- *if $I_k \in \mathcal{K}^s \setminus \mathcal{H}^s$, $P^k_j = 0$ for all $j = 1, \dots, \widetilde{n}_s$.*

Remark 3.15. Property (A1) implies that Step (s-3) is feasible, which, in turn, guarantees that Steps (s-4)-(a) and (s-4)-(b) are well defined, and $\tilde{z}^s$ is a system of coordinates because the differential of the application $y^s \mapsto \tilde{z}^s$ at 0 is equal to the identity map. Property (A4) guarantees that Step (s-5)-(b) is feasible. Property (A2) ensures that, at the end of the algorithm, the system of coordinates z^r is a system of privileged coordinates. Property (A5) finally ensures that for $s = r$, the approximation $\widehat{\xi}$ of ξ is in canonical form.

By Remark 3.15, Theorem 3.11 is a consequence of Proposition 3.12 whose argument goes by induction on s.

Proof of Proposition 3.12. We begin by showing that Properties (A1)-(A5) hold true for $s = 1$.

Claim 1. *The family of vectors* $\{\xi^1_I(a^1)\}_{I\in\mathcal{K}^1}$ *is linearly independent, i.e., Property* (A1) *holds true for* $s = 1$.

Proof of Claim 1. By construction, for every $I \in \mathcal{J}$, one has $\xi^1_I(a^1) = X_I(a)$, which belongs to $\mathbb{R}^n \times \{0\}$. For $i \in \mathcal{G}^1 \setminus \mathcal{J}^1$, the vector $\xi^1_i(a^1)$ belongs to $\mathbb{R}^n \times \mathbb{R}^{\widetilde{k}_1-k_1}$, and the family of vectors $\{\xi^1_i(a^1)\}_{i\in\mathcal{G}^1\setminus\mathcal{J}^1}$ is linearly independent. Therefore, the family of vectors $\{\xi^1_I(a^1)\}_{I\in\mathcal{K}^1}$ is linearly independent and Claim 1 holds true.

□

Claim 2. *For* $j = 1,\dots,\widetilde{n}_1$, *one has* $\mathrm{ord}^1_{a^1}(z^1_j) = 1$, *i.e., Property* (A2) *holds true for* $s = 1$.

Proof of Claim 2. For $j = 1,\dots,\widetilde{n}_1$, one has by construction $\xi^1_j \cdot z^1_j(a^1) = 1$. Thus, one has

$$\mathrm{ord}^1_{a^1}(z^1_j) \leq 1.$$

Since z^1 is a system of coordinates centered at a^1, one has $z^1_j(a^1) = 0$, which implies that

$$\mathrm{ord}^1_{a^1}(z^1_j) > 0.$$

Therefore, one gets $\mathrm{ord}^1_{a^1}(z^1_j) = 1$ and Claim 2 holds true.

□

Claim 3. *For* $I_j \in \mathcal{K}^1$ *with* $|I_j| > 1$, *one has* $\mathrm{ord}^1_{a^1}(z^1_j) > 1$, *i.e., Property* (A3) *holds true for* $s = 1$.

Proof of Claim 3. For $|I_j| \geq 2$, i.e. $I_j \in \mathcal{K}^1 \setminus \mathcal{J}^1$, one computes $\xi^1_k \cdot z^1_j$ at a^1 for every $k \in \{1,\dots,\widetilde{n}_1\}$.

$$\begin{aligned}\xi^1_k \cdot z^1_j(a^1) &= \xi^1_k \cdot y^1_j(a^1) - \sum_{i=1}^{\widetilde{n}_1} (\xi^1_i \cdot y^1_j)(a^1)(\xi^1_k \cdot y^1_i)(a^1)\\ &= \xi^1_k \cdot y^1_j(a^1) - \xi^1_k \cdot y^1_j(a^1) = 0.\end{aligned}$$

Then, by definition, one has $\mathrm{ord}^1_{a^1}(z^1_j) > 1$ for $|I_j| > 1$ and Claim 3 holds true.

□

Claim 4. *For* $i = 1,\dots,m$, *and* $j = 1,\dots,\widetilde{n}_1$, *the* $j^{th}-$*component of* ξ^1_i *in coordinates* z^1 *is equal to* 1 *if* $i = j$, *and equal to* 0 *otherwise. In other words, for* $i = 1,\dots,m$, *the* $\widetilde{n}_1$ *first components of* ξ^1_i *verify Eq. (3.30). Properties* (A4) *and* (A5) *hold true for* $s = 1$.

Proof of Claim 4. By Claim 1, $\xi^1_1(a^1),\dots,\xi^1_{\widetilde{n}_1}(a^1)$ is a basis of $\mathbb{R}^{\widetilde{n}_1}$, and thus the linear change of coordinates y^1 exists. As $\partial_{y^1_j} = \xi^1_j(a^1)$, and $z^1_j = y^1_j$ for $j = 1,\dots,\widetilde{n}_1$, Claim 4 holds true.

□

Therefore, Properties (A1)-(A5) hold true for $s = 1$. Let $1 \leq s \leq r$. Let us now assume that Properties (A1)-(A5) hold true for $1 \leq s' \leq s$. We will show that they still hold true for $s+1$.

Claim 5. *The vector fields* $\{\xi^{s+1}_i\}_{i=1,\dots,m}$ *are well defined. Moreover, one has* $\mathrm{ord}^s_{a^{s+1}}(P_k) = s$.

Proof of Claim 5. Consider $I_k \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}$, then one has $I_k = [I_{k_1}, I_{k_2}]$. By Eq. (3.18), one has

$$P_k(z^s) = \frac{z^s_{k_1}}{\alpha^{k_1}_{k_2} + 1} P_{k_2}(z^s).$$

Since $|I_{k_1}| \leq s$ and $|I_{k_2}| \leq s$, we have $k_1 \leq \widetilde{n}_s$ and $k_2 \leq \widetilde{n}_s$, thus the right-hand side of the above equation is well defined in coordinates $z^s = (z^s_1, \ldots, z^s_{\widetilde{n}_s})$. Therefore, the new vector fields $\{\xi^{s+1}_i\}_{i=1,\ldots,m}$ are also well defined.

Since $\mathrm{ord}^s_{a^{s+1}}(z^s_{k_1} P_{k_2}) = \mathrm{ord}^s_{a^{s+1}}(z^s_{k_1}) + \mathrm{ord}^s_{a^{s+1}}(P_{k_2})$, and by inductive hypothesis (namely (A2) holds true at step s), one has

$$\mathrm{ord}^s_{a^{s+1}}(z^s_{k_1}) = |I_{k_1}|, \text{ and } \mathrm{ord}^s_{a^{s+1}}(P_{k_2}) = |I_{k_2}| - 1.$$

Therefore, one has $\mathrm{ord}^s_{a^{s+1}}(P_k) = |I_{k_1}| + |I_{k_2}| - 1 = s$.

□

Claim 6. *For $I_k \in \mathcal{K}^{s+1}$ with $|I_k| \leq s+1$, one has*

$$\xi^{s+1}_{I_k}(z^s, v^{s+1}) = \xi^s_{I_k}(z^s) + \sum_{I_j \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} \tilde{P}^k_j(z^s) \partial_{v^{s+1}_j}, \tag{3.33}$$

where

$$\tilde{P}^k_j(z^s) = P^k_j(z^s_1, \ldots, z^s_{\widetilde{n}_s}) + \tilde{R}^k_j(z^s),$$

with $\mathrm{ord}^s_{a^{s+1}}(P^k_j) = |I_j| - |I_k|$ *and* $\mathrm{ord}^s_{a^{s+1}}(\tilde{R}^k_j) > |I_j| - |I_k|$.

Proof of Claim 6. The proof goes by induction on the length $|I_k|$. For $|I_k| = 1$, one has (by construction)

$$\xi^{s+1}_k(z^s, v^{s+1}) = \xi^s_k(z^s) + \sum_{\substack{I_j \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1} \\ \phi(j) = k}} P_j(z^s) \partial_{v^{s+1}_j}.$$

By Claim 5, if $\phi(j) = k$, then $\mathrm{ord}^s_{a^{s+1}}(P_j) = s = |I_j| - |I_k|$. Claim 6 holds true for $|I_k| = 1$.

Assume that Claim 6 holds true for every $I \in \mathcal{K}^{s+1}$ of length less than or equal to s_1. Consider $I_k \in \mathcal{K}^{s+1}$ with $|I_k| = s_1 + 1$. In coordinates (z^s, v^{s+1}), one has

$$\begin{aligned}
\xi^{s+1}_{I_k} &= [\xi^{s+1}_{I_{k_1}}, \xi^{s+1}_{I_{k_2}}] \\
&= [\xi^s_{I_{k_1}} + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} (P^{k_1}_i + \tilde{R}^{k_1}_i) \partial_{v^{s+1}_i}, \\
&\quad \xi^s_{I_{k_2}} + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} (P^{k_2}_i + \tilde{R}^{k_2}_i) \partial_{v^{s+1}_i}] \\
&= [\xi^s_{I_{k_1}}, \xi^s_{I_{k_2}}] \\
&\quad + \sum_{I_i \in \mathcal{G}^{s+1} \setminus \mathcal{J}^{s+1}} \{\xi^s_{I_{k_2}} \cdot (P^{k_1}_i + \tilde{R}^{k_1}_i) - \xi^s_{I_{k_1}} \cdot (P^{k_2}_i + \tilde{R}^{k_2}_i)\} \partial_{v^{s+1}_i}.
\end{aligned}$$

Since (A5) holds true up to step s, one has

$$\begin{aligned}
&\xi^s_{I_{k_1}} \cdot (P_i^{k_2} + \tilde{R}_i^{k_2}) \\
= &\left[\sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} + R_j^{k_1})\partial_{z_j^s} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_1} \partial_{z_\ell^s}\right] \cdot (P_i^{k_2} + \tilde{R}_i^{k_2}) \\
= &\sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} + R_j^{k_1})\partial_{z_j^s} P_i^{k_2} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_1} \partial_{z_\ell^s} P_i^{k_2} \\
&+ \sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} + R_j^{k_1})\partial_{z_j^s} \tilde{R}_i^{k_2} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_1} \partial_{z_\ell^s} \tilde{R}_i^{k_2} \\
= &\sum_{I_j \in \mathcal{H}^s} P_j^{k_1} \partial_{z_j^s} P_i^{k_2} + \Bigg[\sum_{I_j \in \mathcal{H}^s} R_j^{k_1} \partial_{z_j^s} P_i^{k_2} + \sum_{I_j \in \mathcal{H}^s} (P_j^{k_1} + R_j^{k_1})\partial_{z_j^s} \tilde{R}_i^{k_2} \\
&+ \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_1} \partial_{z_\ell^s} \tilde{R}_i^{k_2}\Bigg] \\
:= &\sum_{I_j \in \mathcal{H}^s} P_j^{k_1} \partial_{z_j^s} P_i^{k_2} + \mathcal{R}_{i,1}.
\end{aligned}$$

We first show that every term in $\mathcal{R}_{i,1}$ has, at a^{s+1}, an order strictly greater than $s+1-|I_k|$. Indeed, for $I_j \in \mathcal{H}^s$, since

$$\mathrm{ord}^s_{a^{s+1}}(z_j) = |I_j|,\ \mathrm{ord}^s_{a^{s+1}}(P_i^{k_2}) = |I_i| - |I_{k_2}|,$$
$$\text{and } \mathrm{ord}^s_{a^{s+1}}(R_j^{k_1}) > |I_j| - |I_{k_1}|,$$

one has $\mathrm{ord}^s_{a^{s+1}}(R_j^{k_1}\partial_{z_j^s} P_i^{k_2}) > |I_j| - |I_{k_1}| + (|I_i| - |I_{k_2}|) - |I_j| = |I_i| - |I_k|$, with $|I_i| = s+1$.

Note that $\mathrm{ord}^s_{a^{s+1}}((P_j^{k_1} + R_j^{k_1})\partial_{z_j^s} \tilde{R}_i^{k_2}) = \mathrm{ord}^s_{a^{s+1}}(P_j^{k_1}\partial_{z_j^s} \tilde{R}_i^{k_2})$. Since

$$\mathrm{ord}^s_{a^{s+1}}(P_j^{k_1}) = |I_j| - |I_{k_1}|, \text{ and } \mathrm{ord}^s_{a^{s+1}}(\tilde{R}_i^{k_2}) > |I_i| - |I_{k_1}|,$$

then, one has

$$\mathrm{ord}^s_{a^{s+1}}(P_j^{k_1}\partial_{z_j^s} \tilde{R}_i^{k_2}) > |I_j| - |I_{k_2}| + |I_i| - |I_{k_1}| - |I_j| = |I_i| - |I_k|.$$

Recall that, by definition, all the functions have positive order. Therefore, one gets

$$\mathrm{ord}^s_{a^{s+1}}(Q_\ell^{k_1}\partial_{z_\ell^s} \tilde{R}_i^{k_2}) \geq \mathrm{ord}^s_{a^{s+1}}(Q_\ell^{k_1}) > s - |I_{k_1}| \geq s + 1 - |I_k|.$$

In conclusion, one gets

$$\mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,1}) > s + 1 - |I_k|.$$

A similar computation shows that

$$\begin{aligned}
&\xi^s_{I_{k_2}} \cdot (P_i^{k_1} + \tilde{R}_i^{k_1}) \\
= &\sum_{I_j \in \mathcal{H}^s} P_j^{k_2} \partial_{z_j^s} P_i^{k_1} + \Bigg[\sum_{I_j \in \mathcal{H}^s} R_j^{k_2} \partial_{z_j^s} P_i^{k_1} \\
&+ \sum_{I_j \in \mathcal{H}^s} (P_j^{k_2} + R_j^{k_2})\partial_{z_j^s} \tilde{R}_i^{k_1} + \sum_{I_\ell \in \mathcal{K}^s \setminus \mathcal{H}^s} Q_\ell^{k_2} \partial_{z_\ell^s} \tilde{R}_i^{k_1}\Bigg] \\
:= &\sum_{I_j \in \mathcal{H}^s} P_j^{k_2} \partial_{z_j^s} P_i^{k_1} + \mathcal{R}_{i,2}, \text{ with } \mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,2}) > s + 1 - |I_k|.
\end{aligned}$$

Therefore, one gets

$$\begin{aligned}\xi^{s+1}_{I_k} &= \xi^s_{I_k} + \sum_{I_i\in\mathcal{G}^{s+1}\setminus\mathcal{J}^{s+1}} \{\sum_{I_j\in\mathcal{H}^s} (P^{k_1}_j \partial_{z^s_j} P^{k_2}_i - P^{k_2}_j \partial_{z^s_j} P^{k_1}_i)\}\partial_{v_i} \\ &\quad + \sum_{I_i\in\mathcal{G}^{s+1}\setminus\mathcal{J}^{s+1}} (\mathcal{R}_{i,1}+\mathcal{R}_{i,2})\partial_{v_i},\end{aligned}$$

with

$$\mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,1}+\mathcal{R}_{i,2}) \geq \min\left(\mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,1}),\ \mathrm{ord}^s_{a^{s+1}}(\mathcal{R}_{i,2})\right) > s+1-|I_k|.$$

Since

$$\sum_{I_j\in\mathcal{H}^s} (P^{k_1}_j \partial_{z^s_j} P^{k_2}_i - P^{k_2}_j \partial_{z^s_j} P^{k_1}_i) = P^k_i,$$

and $\mathrm{ord}^s_{a^{s+1}}(P^k_i) = |I_i| - |I_k|$ by Corollary 3.9, one gets

$$\xi^{s+1}_{I_k}(z^s, v^{s+1}) = \xi^s_{I_k}(z^s) + \sum_{I_i\in\mathcal{G}^{s+1}\setminus\mathcal{J}^{s+1}} (P^k_i(z^s) + \tilde{R}^k_i(z^s))\partial_{v_i},$$

with $\mathrm{ord}^s_{a^{s+1}}(P^k_i) = s+1-|I_k|$ and $\mathrm{ord}^s_{a^{s+1}}(\tilde{R}^k_i) > s+1-|I_k|$.

Therefore, Claim 6 still holds true for $I_k \in \mathcal{K}^{s+1}$ with $|I_k| = s_1+1$. This terminates the induction, and Claim 6 is now proved.

□

Claim 7. *The family of vectors* $\{\xi^{s+1}_{I_k}(a^{s+1})\}_{I_k\in\mathcal{K}^{s+1}}$ *is linearly independent, i.e.,* (A1) *holds true at Step* $s+1$.

Proof of Claim 7. Claim 6 implies that for every $I_k \in \mathcal{K}^s$, one has

$$\xi^{s+1}_{I_k}(a^{s+1}) = \xi^s_{I_k}(a^s) \in \mathbb{R}^{\widetilde{n}_s} \times \{0\}.$$

Corollary 3.9 implies that for every $I_k \in \mathcal{G}^{s+1}\setminus\mathcal{J}^{s+1}$, one has

$$\xi^{s+1}_{I_k}(a^{s+1}) = \xi^s_{I_k}(a^s) + \partial_{v_k} \in \mathbb{R}^{\widetilde{n}_s} \times \mathbb{R}^{\widetilde{k}_{s+1}-k_{s+1}}.$$

Therefore, by (A1) at step s, the vectors $\{\xi^{s+1}_{I_k}(a^{s+1})\}_{I_k\in\mathcal{K}^{s+1}}$ are linearly independent.

□

Claim 8. *After performing* (s+1)-4-(a) *and* (s+1)-4-(b) *in the Desingularization Algorithm, one has, for every* $I_j \in \mathcal{H}^{s+1}$,

$$\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}^{s+1}_j) = |I_j|,$$

and for every $I_j \in \mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}$,

$$\mathrm{ord}^{s+1}_{a^{s+1}}(z^{s+1}_j) > s+1.$$

The proof of Claim 8 is based on the following result due to Bellaïche [11, Lemma 4.12].

Lemma 3.13. *Let* $\{X_1,\dots,X_m\}$ *be a family vector fields defined on* Ω. *Consider* $\{W_1,\dots,W_n\}$ *a frame adapted to the flag* $L^1(a)\subset\cdots\subset L^r(a) = \mathbb{R}^n$ *at* $a\in\Omega$ *(cf.* Remark 3.4*). A function* f *is of order strictly greater than* s *at* a *is and only if*

$$(W_1^{\alpha_1}\cdots W_n^{\alpha_n} f)(a) = 0,$$

for all $\alpha = (\alpha_1,\dots,\alpha_n)$ *such that* $w(\alpha)\leq s$.

Proof of Claim 8. Claim 7 guarantees that $\{\xi^{s+1}_{I_k}\}_{I_k\in\mathcal{H}^{s+1}}$ is a basis adapted to the flag

$$L^1(a^{s+1})\subset\cdots\subset L^{s+1}(a^{s+1}).$$

Complete $\{\xi^{s+1}_{I_k}\}_{I_k\in\mathcal{H}^{s+1}}$ by other elements of the Lie algebra generated by $\{\xi^{s+1}_i\}_{i=1,\dots,m}$ in order to get a basis adapted to the flag

$$L^1(a^{s+1})\subset\cdots\subset L^{s+1}(a^{s+1})\subset\cdots\subset L^r(a^{s+1}).$$

For $I_j\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}$, Formula (s+1)-4-(b) ensures that

$$((\xi^{s+1}_{I_1})^{\beta_1}\cdots(\xi^{s+1}_{I_{\widetilde{n}_{s+1}}})^{\beta_{\widetilde{n}_{s+1}}}\cdot\tilde{z}^{s+1}_j)(a^{s+1})=0,$$

for all $\beta=(\beta_1,\dots,\beta_{\widetilde{n}_{s+1}})$ such that $w(\beta)\leq s+1$. By Lemma 3.13, one has

$$\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}^{s+1}_j)>s+1,\quad \text{for } I_j\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}.$$

For $I_j\in\mathcal{H}^{s+1}$, Formula (s+1)-4-(a) implies that

$$((\xi^{s+1}_{I_1})^{\beta_1}\cdots(\xi^{s+1}_{I_{j-1}})^{\beta_{j-1}}\cdot\tilde{z}^{s+1}_j)(a^{s+1})=0,$$

for all $\beta=(\beta_1,\dots,\beta_{j-1})$ such that $w(\beta)\leq|I_j|-1$. Using again Lemma 3.13, one has

$$\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}^{s+1}_j)>|I_j|-1,\quad \text{for } I_j\in\mathcal{H}^{s+1}.$$

By construction, one already has that $\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}^{s+1}_j)\leq\widetilde{w}_j=|I_j|$. Therefore, one finally gets

$$\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}^{s+1}_j)=|I_j|,\quad \text{for } I_j\in\mathcal{H}^{s+1}.$$

Claim 8 is now proved.

□

Claim 9. *The change of coordinates* $(\Psi^{s+1}_j)_{j=1,\dots,\widetilde{n}_{s+1}}$ *is well defined, i.e., Property* (A4) *holds true.*

Proof of Claim 9. After performing Steps (s+1)-4-(a) and (s+1)-4-(b), one obtains a new system of coordinates $\tilde{z}^{s+1}$. In this system of coordinates, one can write ξ^{s+1}_i as

$$\begin{aligned}\xi^{s+1}_i(\tilde{z}^{s+1}) &= \partial_{\tilde{z}^{s+1}_i}+\sum_{\substack{I_j\in\mathcal{H}^{s+1}\\|I_j|\geq 2}}(\tilde{P}_{i,j}(\tilde{z}^{s+1})+\tilde{R}_{i,j}(\tilde{z}^{s+1}))\partial_{\tilde{z}^{s+1}_j}\\&\quad+\sum_{I_\ell\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}}\tilde{Q}_{i,\ell}(\tilde{z}^{s+1})\partial_{\tilde{z}^{s+1}_\ell},\end{aligned}$$

where $\tilde{P}_{i,j}$, $\tilde{R}_{i,j}$, and $\tilde{Q}_{i,\ell}$ are polynomials with $\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{P}_{i,j})=\widetilde{w}_j-1$, $\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{R}_{i,j})\geq\widetilde{w}_j$, and $\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{Q}_{i,\ell})>s$.

Since

$$\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}^{s+1}_j)=\widetilde{w}_j,\quad \text{for } I_j\in\mathcal{H}^{s+1},$$

$$\mathrm{ord}^{s+1}_{a^{s+1}}(\tilde{z}^{s+1}_j)>s+1,\quad \text{for } I_j\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1},$$

the polynomials $\tilde{P}_{i,j}$ contain only variables $\tilde{z}^{s+1}_k$ with $\widetilde{w}_k\leq\widetilde{w}_j-1$.

Let us now show that there exists a change of coordinates Ψ^{s+1} which transforms coordinates $\tilde{z}^{s+1}$ into new coordinates z^{s+1} such that

$$\mathrm{ord}^{s+1}_{a^{s+1}}(z^{s+1}_j) = \widetilde{w}_j, \quad \text{for } I_j \in \mathcal{H}^{s+1},$$

$$\mathrm{ord}^{s+1}_{a^{s+1}}(z^{s+1}_j) > s+1, \quad \text{for } I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1},$$

and in the new coordinates, the $\widetilde{n}_{s+1}$ first components $\xi^{s+1}_{i,j}(z^{s+1})$ of $\xi^{s+1}_i(z^{s+1})$ are in the form

$$\xi^{s+1}_{i,j}(z^{s+1}) = \delta_{i,\phi(j)} P_j(z^{s+1}_1, \ldots, z^{s+1}_{j-1}) + R_{i,j}(z^{s+1}), \quad j = 1, \ldots, \widetilde{n}_{s+1},$$

with $\mathrm{ord}^{s+1}_{a^{s+1}}(R_{i,j}) \geq \widetilde{w}_j$.

We first note that, once one has

$$\mathrm{ord}^{s+1}_{a^{s+1}}(z^{s+1}_j) = \widetilde{w}_j \quad \text{for } I_j \in \mathcal{H}^{s+1},$$

and

$$\mathrm{ord}^{s+1}_{a^{s+1}}(z^{s+1}_j) > s+1 \quad \text{for } I_j \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1},$$

then, the order of $P_{i,j}$ will be equal to its weighted degree, and thus automatically equal to $\widetilde{w}_j - 1$ by construction of these polynomials. Consider now $\check{\xi}_i^{s+1}$ defined in coordinates $\tilde{z}^{s+1}$ by

$$\check{\xi}^{s+1}_i(\tilde{z}^{s+1}) = \partial_{\tilde{z}^{s+1}_i} + \sum_{I_j \in \mathcal{H}^{s+1}} \tilde{P}_{i,j}(\tilde{z}^{s+1}) \partial_{\tilde{z}^{s+1}_j}.$$

Recall that, by construction, the vector fields $\{\check{\xi}_i\}_{i=1,\ldots,m}$ generate a free nilpotent Lie algebra of step $s+1$. Moreover, in the canonical coordinates of the second kind $(z^{s+1}_1, \ldots, z^{s+1}_{\widetilde{n}_{s+1}})$ associated with $\{\check{\xi}^{s+1}_{I_k}\}_{I_k \in \mathcal{H}^{s+1}}$, the vector fields $\check{\xi}^{s+1}_i$ are in the canonical form, i.e.

$$\check{\xi}^{s+1}_i = \partial_{\tilde{z}^{s+1}_i} + \sum_{\substack{I_j \in \mathcal{H}^{s+1} \\ \phi(j)=i}} P_j(z^{s+1}) \partial_{z^{s+1}_j}.$$

By definition of a system of coordinates, there exist $\widetilde{n}_{s+1}$ smooth functions $(\Psi^{s+1}_1, \ldots, \Psi^{s+1}_{\widetilde{n}_{s+1}})$ such that, for $j = 1, \ldots, \widetilde{n}_{s+1}$, one has

$$z^{s+1}_j = \Psi^{s+1}_j(\tilde{z}^{s+1}_1, \ldots, \tilde{z}^{s+1}_{\widetilde{n}_{s+1}}).$$

Expand now Ψ^{s+1}_j in Taylor series. Since $\mathrm{ord}^{s+1}_{a^{s+1}}(z^{s+1}) = \widetilde{w}_j$, the Taylor expansion of Ψ^{s+1}_j is a *polynomial* of weighted degree equal to $\widetilde{w}_j$. Claim 9 is now proved.

□

Remark 3.16. The change of coordinates $(\Psi^{s+1}_j)_{j=1,\ldots,\widetilde{n}_{s+1}}$ is computed by identification. Indeed, since $\mathrm{ord}^{s+1}_{a^{s+1}}(z^{s+1}_j) = \widetilde{w}_j$, and the nonholonomic order does not depend on any system of coordinates, then Ψ^{s+1}_j is a function of order $\widetilde{w}_j$ at a^{s+1}, i.e., the Taylor expansion of Ψ^{s+1}_j at a^{s+1} contains only monomials of weighted degree equal to $\widetilde{w}_j$, and there is a finite number of such monomials. Therefore, the function Ψ^{s+1}_j is necessarily in the following form

$$\Psi^{s+1}_j(\tilde{z}^{s+1}) = \sum_{w(\alpha)=\widetilde{w}_j} \beta^\alpha_j (\tilde{z}^{s+1}_1)^{\alpha_1} \ldots (\tilde{z}^{s+1}_{\widetilde{n}_{s+1}})^{\alpha_{\widetilde{n}_{s+1}}}, \tag{3.34}$$

where β^α_j are real numbers.

Eq. (3.34) is a finite sum and therefore the scalar coefficients (φ_j^α) can be obtained by identification. Claim 9 guarantees that such a set of real numbers (φ_j^α) exists. Note also that, due to the constraint on the weight, Eq. (3.34) only involves variables $\tilde{z}_k^{s+1}$ of weight less than $\widetilde{w}_j$, implying that the change of coordinates $(\Psi_j^{s+1})_{j=1,\dots,\widetilde{n}_{s+1}}$ is naturally triangular.

Remark 3.17. Let us now illustrate Remark 3.16 with a simple example. Consider here a nilpotent system of step 2 generated by two vector fields (ξ_1, ξ_2). We have $\xi_{I_1} = \xi_1$, $\xi_{I_2} = \xi_2$ and $\xi_{I_3} = [\xi_1, \xi_2]$. In coordinates $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$, ξ_1 and ξ_2 are necessarily in the form $\xi_1 = (1, 0, \alpha_1\tilde{z}_1 + \alpha_2\tilde{z}_2)$, and $\xi_2 = (0, 1, \beta_1\tilde{z}_1 + \beta_2\tilde{z}_2)$, where α_1, α_2, β_1 and β_2 are real numbers verifying $\beta_1 - \alpha_2 = 1$. As mentioned in Remark 3.16, in the change of coordinates (Ψ_1, Ψ_2, Ψ_3), every Ψ_j is a homogeneous polynomial of weighted degree equal to $\widetilde{w}_j$. Set

$$z = (\Psi_1(\tilde{z}), \Psi_2(\tilde{z}), \Psi_3(\tilde{z})) =: (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 + a\tilde{z}_1\tilde{z}_2 + b\tilde{z}_1^2 + c\tilde{z}_2^2),$$

with a, b, and c to be determined. One imposes that $\xi_2(z) = (0, 1, z_1)$. After computation, one gets

$$\begin{aligned} (\alpha_1 + 2b)\tilde{z}_1 + (\alpha_2 + a)\tilde{z}_2 &= 0, \\ (\beta_1 + a)\tilde{z}_1 + (\beta_2 + 2c)\tilde{z}_2 &= z_1 = \tilde{z}_1. \end{aligned}$$

By identification, one gets $a = -\alpha_2$, $b = -\frac{\alpha_1}{2}$, $c = -\frac{\beta_2}{2}$, and in that case, $\beta_1 + a = \beta_1 - \alpha_2 = 1$ is automatically verified. Then, the triangular change of coordinates

$$(z_1, z_2, z_3) = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 - \alpha_2\tilde{z}_1\tilde{z}_2 - \frac{\alpha_1}{2}\tilde{z}_1^2 - \frac{\beta_2}{2}\tilde{z}^2)$$

puts ξ_1 and ξ_2 into the canonical form.

Claim 10. *Property* (A5) *holds true at step* $s + 1$.

Proof of Claim 10. The proof goes by induction on the length of $I_k \in \mathcal{K}^{s+1}$. It is similar to the one of Claim 6.

For $|I_k| = 1$, one has

$$\begin{aligned} \xi_i^{s+1}(z^{s+1}) &= \sum_{\substack{I_j \in \mathcal{H}^{s+1} \\ \phi(j)=i}} (P_j(z^{s+1}) + R_{i,j}(z^{s+1}))\partial_{z_j^{s+1}} \\ &\quad + \sum_{I_\ell \in \mathcal{K}^{s+1} \setminus \mathcal{H}^{s+1}} Q_{i,\ell}(z^{s+1})\partial_{z_\ell^{s+1}}, \end{aligned}$$

with $\mathrm{ord}_{a^{s+1}}^{s+1}(P_j) = |I_j| - 1$, $\mathrm{ord}_{a^{s+1}}^{s+1}(R_{i,j}) > |I_j| - 1$, $\mathrm{ord}_{a^{s+1}}^{s+1}(Q_{i,\ell}) > s$. Claim 10 holds true for $|I_k| = 1$.

Assume that Claim 10 holds for brackets of length less than s_1. We show that it still holds

true for brackets of length s_1+1. Consider $I_k \in \mathcal{K}^{s+1}$ with $|I_k| = s_1+1$. Then, one has

$$\begin{aligned}
&\xi^{s+1}_{I_k} = [\xi^{s+1}_{I_{k_1}}, \xi^{s+1}_{I_{k_2}}] \\
= \;& [\sum_{I_j\in\mathcal{H}^{s+1}} (P_j^{k_1}+R_j^{k_1})\partial_{z_j^{s+1}} + \sum_{I_\ell\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}} Q_\ell^{k_1}\partial_{z_\ell^{s+1}}, \\
& \sum_{I_j\in\mathcal{H}^{s+1}} (P_j^{k_2}+R_j^{k_2})\partial_{z_j^{s+1}} + \sum_{I_\ell\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}} Q_\ell^{k_2}\partial_{z_\ell^{s+1}}] \\
= \;& \sum_{I_j\in\mathcal{H}^{s+1}} [\sum_{I_i\in\mathcal{H}^{s+1}} P_i^{k_1}\partial_{z_i^{s+1}}P_j^{k_2} - P_i^{k_2}\partial_{z_i^{s+1}}P_j^{k_1}]\partial_{z_j^{s+1}} \\
+ \;& \sum_{I_j\in\mathcal{H}^{s+1}} [\sum_{I_i\in\mathcal{H}^{s+1}} \{R_i^{k_1}\partial_{z_i^{s+1}}(P_j^{k_2}+R_j^{k_2}) - R_i^{k_2}\partial_{z_i^{s+1}}(P_j^{k_1}+R_j^{k_1})\} \\
& +\{P_i^{k_1}\partial_{z_i^{s+1}}R_j^{k_2} - P_i^{k_2}\partial_{z_i^{s+1}}R_j^{k_1}\} \\
& + \sum_{I_\ell\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}} Q_\ell^{k_1}\partial_{z_\ell^{s+1}}(P_j^{k_2}+R_j^{k_2}) - Q_\ell^{k_2}\partial_{z_\ell^{s+1}}(P_j^{k_1}+R_j^{k_1})]\partial_{z_j^{s+1}} \\
+ \;& \sum_{I_j\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}} [\sum_{I_i\in\mathcal{H}^{s+1}} (P_i^{k_1}+R_i^{k_1})\partial_{z_i^{s+1}}Q_j^{k_2} - (P_i^{k_2}+R_i^{k_2})\partial_{z_i^{s+1}}Q_j^{k_1} \\
& + \sum_{I_\ell\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}} Q_\ell^{k_1}\partial_{z_\ell^{s+1}}Q_j^{k_2} - Q_\ell^{k_2}\partial_{z_\ell^{s+1}}Q_j^{k_1}]\partial_{z_j^{s+1}}.
\end{aligned}$$

By the inductive hypothesis, one can proceed as follows.

– Taking into account the relation

$$\mathrm{ord}^{s+1}_{a^{s+1}}(R_i^{k_1}) > |I_i| - |I_{k_1}| \text{ and } \mathrm{ord}^{s+1}_{a^{s+1}}(\partial_{z_i^{s+1}}(P_j^{k_2}+R_j^{k_2})) \geq |I_j| - |I_{k_2}| - |I_i|,$$

one deduces that

$$\mathrm{ord}^{s+1}_{a^{s+1}} R_i^{k_1}\partial_{z_i^{s+1}}(P_j^{k_2}+R_j^{k_2}) > |I_j| - |I_k|.$$

By a similar argument, $\mathrm{ord}^{s+1}_{a^{s+1}} R_i^{k_2}\partial_{z_i^{s+1}}(P_j^{k_1}+R_j^{k_1}) > |I_j| - |I_k|$. Therefore, one gets

$$\mathrm{ord}^{s+1}_{a^{s+1}}(R_i^{k_1}\partial_{z_i^{s+1}}(P_j^{k_2}+R_j^{k_2}) - R_i^{k_2}\partial_{z_i^{s+1}}(P_j^{k_1}+R_j^{k_1})) > |I_j| - |I_k|.$$

– Since

$$\mathrm{ord}^{s+1}_{a^{s+1}}(P_i^{k_1}) = |I_i| - |I_{k_1}| \text{ and } \mathrm{ord}^{s+1}_{a^{s+1}}(\partial_{z_i^{s+1}}R_j^{k_2}) > |I_j| - |I_{k_2}| - |I_i|,$$

then

$$\mathrm{ord}^{s+1}_{a^{s+1}}(P_i^{k_1}\partial_{z_i^{s+1}}R_j^{k_2}) > |I_j| - |I_k|.$$

By a similar argument, $\mathrm{ord}^{s+1}_{a^{s+1}}(P_i^{k_2}\partial_{z_i^{s+1}}R_j^{k_1}) > |I_j| - |I_k|$. One thus obtains

$$\mathrm{ord}^{s+1}_{a^{s+1}}(P_i^{k_1}\partial_{z_i^{s+1}}R_j^{k_2} - P_i^{k_2}\partial_{z_i^{s+1}}R_j^{k_1}) > |I_j| - |I_k|.$$

– Using the fact that

$$\mathrm{ord}^{s+1}_{a^{s+1}}(\partial_{z_\ell^{s+1}}(P_j^{k_2}+R_j^{k_2})) > |I_j| - |I_{k_2}| - (s+1) \text{ and } \mathrm{ord}^{s+1}_{a^{s+1}}(Q_\ell^{k_1}) > s+1-|I_{k_1}|,$$

then

$$\mathrm{ord}^{s+1}_{a^{s+1}}(Q_\ell^{k_1}\partial_{z_\ell^{s+1}}(P_j^{k_2}+R_j^{k_2})) > |I_j| - |I_k|.$$

By a similar argument, $\mathrm{ord}^{s+1}_{a^{s+1}} Q_\ell^{k_2}\partial_{z_\ell^{s+1}}(P_j^{k_1}+R_j^{k_1} > |I_j|-|I_k|$.
One deduces

$$\mathrm{ord}^{s+1}_{a^{s+1}}(Q_\ell^{k_1}\partial_{z_\ell^{s+1}}(P_j^{k_2}+R_j^{k_2}) - Q_\ell^{k_2}\partial_{z_\ell^{s+1}}(P_j^{k_1}+R_j^{k_1})) > |I_j|-|I_k|.$$

– Recall that

$$\mathrm{ord}^{s+1}_{a^{s+1}}(P_i^{k_1}+R_i^{k_1}) = |I_i|-|I_{k_1}| \text{ and } \mathrm{ord}^{s+1}_{a^{s+1}}(\partial_{z_i^{s+1}}Q_j^{k_2}) > s+1-|I_{k_2}|-|I_i|,$$

then

$$\mathrm{ord}^{s+1}_{a^{s+1}}((P_i^{k_1}+R_i^{k_1})\partial_{z_i^{s+1}}Q_j^{k_2}) > s+1-|I_k|.$$

By a similar argument,

$$\mathrm{ord}^{s+1}_{a^{s+1}}((P_i^{k_2}+R_i^{k_2})\partial_{z_i^{s+1}}Q_j^{k_1}) > s+1-|I_k|.$$

Therefore, it yields

$$\mathrm{ord}^{s+1}_{a^{s+1}}((P_i^{k_1}+R_i^{k_1})\partial_{z_i^{s+1}}Q_j^{k_2} - (P_i^{k_2}+R_i^{k_2})\partial_{z_i^{s+1}}Q_j^{k_1}) > s+1-|I_k|.$$

– Since $\partial_{z_\ell^{s+1}}Q_j^{k_2}$ is a function, one knows by definition that
$\mathrm{ord}^{s+1}_{a^{s+1}}(\partial_{z_\ell^{s+1}}Q_j^{k_2}) \geq 0$. As $\mathrm{ord}^{s+1}_{a^{s+1}}(Q_\ell^{k_1}) > s+1-|I_{k_1}|$, one has

$$\mathrm{ord}^{s+1}_{a^{s+1}}(Q_\ell^{k_1}\partial_{z_\ell^{s+1}}Q_j^{k_2}) > s+1-|I_{k_1}| = s+1-(|I_k|-|I_{k_2}|) > s+1-|I_k|.$$

By a similar argument, $\mathrm{ord}^{s+1}_{a^{s+1}}(Q_\ell^{k_2}\partial_{z_\ell^{s+1}}Q_j^{k_1}) > s+1-|I_k|$. One hence derives

$$\mathrm{ord}^{s+1}_{a^{s+1}}(Q_\ell^{k_1}\partial_{z_\ell^{s+1}}Q_j^{k_2} - Q_\ell^{k_2}\partial_{z_\ell^{s+1}}Q_j^{k_1}) > s+1-|I_k|.$$

Summing up the above terms, one gets, for $I_k \in \mathcal{K}^{s+1}$ of length s_1+1, that the bracket $\xi^{s+1}_{I_k}$ can be written in the form

$$\begin{aligned}\xi^{s+1}_{I_k}(z^{s+1}) &= \sum_{I_j\in\mathcal{H}^{s+1}} (P_j^k(z^{s+1})+R_j^k(z^{s+1}))\partial_{z_j^{s+1}} \\ &\quad + \sum_{I_j\in\mathcal{K}^{s+1}\setminus\mathcal{H}^{s+1}} Q_j^k(z^{s+1})\partial_{z_j^{s+1}},\end{aligned}$$

with

$$\mathrm{ord}^{s+1}_{a^{s+1}}(P_j^k) = |I_j|-|I_k|, \qquad \mathrm{ord}^{s+1}_{a^{s+1}}(R_j^k) > |I_j|-|I_k|,$$

and $\mathrm{ord}^{s+1}_{a^{s+1}}(Q_\ell^k) > s+1-|I_k|$.
Claim 10 is now proved.

□

In conclusion, Properties (A1)-(A5) still hold true at step $s+1$ in the Desingularization Algorithm. The induction step is established, which terminates the proof of Proposition 3.12.

□

3.4 Global Steering Method for Regular Systems

By taking into account the Desingularization Algorithm presented in Chapter 3.3, we assume in this chapter and without loss of generality that the family of vectors fields $X = \{X_1, \dots, X_m\}$ is free up to step r (cf. Definition 3.25). Recall that, in that case, every point $x \in \Omega$ is regular and the growth vector is constant on Ω. We present in Section 3.4.1 an algebraic construction of privileged coordinates and a nonholonomic first order approximation of X under canonical form. For regular systems, this construction also provides a continuously varying system of privileged coordinates. We then propose in Section 3.4.2 a global motion planning algorithm for regular systems.

3.4.1 Construction of the approximate system $\mathcal{A}^X$

For every point a in Ω, we construct the first order approximate system $\mathcal{A}^X(a)$ of the system X at a (cf. Definition 3.14) as follows :

Step (1) Take $\{X_{I_j}\}_{I_j \in \mathcal{H}^r}$. Set $w_j = \widetilde{w}_j$ for $j = 1, \dots, n$.

Step (2) Construct the linear system of coordinates $y = (y_1, \dots, y_n)$ such that $\partial_{y_j} = X_{I_j}(a)$.

Step (3) Build the system of privileged coordinates $\widetilde{z} = (\widetilde{z}_1, \dots, \widetilde{z}_n)$ by the following iterative formula : for $j = 1, \dots, n$,

$$\widetilde{z}_j := y_j + \sum_{k=2}^{w_j - 1} h_k(y_1, \dots, y_{j-1}), \tag{3.35}$$

where, for $k = 2, \dots, w_j - 1$,

$$h_k(y_1, \dots, y_{j-1}) = - \sum_{\substack{|\alpha|=k \\ w(\alpha)<w_j}} X_{I_1}^{\alpha_1} \dots X_{I_{j-1}}^{\alpha_{j-1}} \cdot (y_j + \sum_{q=2}^{k-1} h_q)(y)|_{y=0} \ \frac{y_1^{\alpha_1}}{\alpha_1!} \cdots \frac{y_{j-1}^{\alpha_{j-1}}}{\alpha_{j-1}!},$$

with $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Step (4) For $i = 1, \dots, m$, compute the Taylor expansion of $X_i(\widetilde{z})$ at 0, and express every vector field as a sum of vector fields which are homogeneous with respect to the weighted degree defined by the sequence $(w_j)_{j=1,\dots,n}$:

$$X_i(\widetilde{z}) = X_i^{(-1)}(\widetilde{z}) + X_i^{(0)}(\widetilde{z}) + \cdots,$$

where we use $X_i^{(k)}(\widetilde{z})$ to denote the sum of all the terms of weighted degree equal to k. Set $\widehat{X}_i^a(\widetilde{z}) := X_i^{(-1)}(\widetilde{z})$.

Step (5) For $j = 1, \dots, n$, identify homogeneous polynomials Ψ_j of weighted degree equal to w_j such that, in the system of privileged coordinates $z := (z_1, \dots, z_n)$ defined by

$$z_j := \Psi_j(\widetilde{z}_1, \dots, \widetilde{z}_{j-1}), \quad \text{for } j = 1, \dots, n,$$

the approximate system

$$\widehat{X}^a(z) = \{z_* \widehat{X}_1^a(\widetilde{z}), \dots, z_* \widehat{X}_m^a(\widetilde{z})\}$$

is in the canonical form.

Step (6) Set $\mathcal{A}^X(a) := \widehat{X}^a$ and $\Phi^X(a, \cdot) :=$ the mapping $x \mapsto z$.

Remark 3.18. Steps (1)-(3) construct a system of privileged coordinates $\widetilde{z}$. The proof that $\widetilde{z}$ is a system of privileged coordinates is essentially based on Lemma 3.13. Roughly speaking, the idea to obtain $\widetilde{z}_j$ from y_j goes as follows : for every $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $w(\alpha) < w_j$ (so $\alpha_j = \cdots = \alpha_n = 0$), compute $X_{I_1}^{\alpha_1} \cdots X_{I_{j-1}}^{\alpha_{j-1}} \cdot y_j(y)|_{y=0}$. If it is not equal to zero, then replace y_j by

$$y_j - (X_{I_1}^{\alpha_1} \cdots X_{I_{j-1}}^{\alpha_{j-1}} \cdot y_j)(y)|_{y=0} \, \frac{y_1^{\alpha_1}}{\alpha_1!} \cdots \frac{y_{j-1}^{\alpha_{j-1}}}{\alpha_{j-1}!}.$$

With that new value of y_j, one gets $X_{I_1}^{\alpha_1} \cdots X_{I_{j-1}}^{\alpha_{j-1}} \cdot y_j(y)|_{y=0} = 0$. Therefore, by Lemma 3.13, one has $\mathrm{ord}_a(\widetilde{z}_j) \geq w_j$ for $j = 1, \ldots, n$. On the other hand, since Step (3) of the construction does not modify the linear part, the system of coordinates $\widetilde{z}$ remains adapted. By Remark 3.5, one also has $\mathrm{ord}_a(\widetilde{z}_j) \leq w_j$, and therefore, $\mathrm{ord}_a(\widetilde{z}_j) = w_j$.

Remark 3.19. The existence of Ψ_j in Step (5) is guaranteed by a simple modification of Claim 9, page 51, see also Remarks 3.16 and 3.17. The key point is, in the current case, the exponential coordinates are *algebraic*.

Remark 3.20. We will propose in Section 3.5 an effective and exact method for steering general nilpotent systems given in the canonical form.

It results from [11] that, for *regular systems*, the mapping

$$\Phi^X : (a, x) \to z$$

is a continuously varying system of privileged coordinates on Ω. Note also that the coordinates z are obtained from y by expressions of the form

$$\begin{aligned} z_1 &= y_1, \\ z_2 &= y_2 + \mathrm{pol}_2(y_1), \\ &\vdots \\ z_n &= y_n + \mathrm{pol}_n(y_1, \ldots, y_{n-1}), \end{aligned}$$

where, for $j = 1, \ldots, n$, the function $\mathrm{pol}_j(\cdot)$ is a polynomial which does not contain constant nor linear terms. Due to the triangular form of this change of coordinates, the inverse change of coordinates from z to y bears exactly the same form. Therefore, the mapping $z = \Phi^X(a, \cdot)$ is defined on the whole Ω, i.e., Φ^X has an infinite injectivity radius. We also note that, by construction, $\mathcal{A}^X$ is a nonholonomic first order approximation (cf. Definition 3.14) and its continuity results from the continuity of the mapping $\Phi^X : (a, x) \mapsto z$. In summary, we have the following proposition.

Proposition 3.14. *The mapping Φ^X is a continuously varying system of privileged coordinates on Ω and the mapping $\mathcal{A}^X$ is a continuous approximation of X on Ω.*

The following theorem is a consequence of Proposition 3.14 and Corollary 3.6.

Theorem 3.15. *Let $\mathcal{V}^c$ be a compact subset of Ω. If $\mathcal{A}^X$ is provided with a sub-optimal steering law* (cf. Definitions 3.16 and 3.17)*, then the LAS method* AppSteer *associated with $\mathcal{A}^X$ and its steering law* (cf. Definition 3.18) *is uniformly locally contractive on $\mathcal{V}^c$.*

Remark 3.21. Due to Step (5) in the construction procedure, the approximate system $\mathcal{A}^X(a)$ is under canonical form in a system of privileged coordinates z. Therefore, $\mathcal{A}^X(a)$ has always the same form, regardless of the control system X or the approximate point $a \in \Omega$. The specificity of each system or each approximate point is hidden in the change of coordinates Φ^X.

Remark 3.22. It is important to notice that the approximate system used in the LAS method is a nonholonomic first order approximation at the *goal* point a (cf. Definition 3.18). Therefore, the steering control always displaces $\mathcal{A}^X(a)$ from some position (which is the image by $\Phi^X(a,\cdot)$ of the current point of the original system) to 0 (which is $\Phi^X(a,a)$ by construction) in coordinates z. The latter fact plays a crucial role in getting the *sub-optimality* for the steering law (see Section 3.5.3 for more details).

3.4.2 Approximate steering algorithm

Let $\mathcal{V}^c \subset \Omega$ be a connected compact set equal to the closure of its interior and $(x^{\text{initial}}, x^{\text{final}}) \in \mathcal{V}^c \times \mathcal{V}^c$. We devise, under the assumptions of Theorem 3.15, an algorithm (Algorithm 6 below) which steers System (4.5) from x^{initial} to x^{final}. That algorithm does not require any a priori knowledge on the critical distance $\varepsilon_{\mathcal{V}^c}$. Note that this algorithm bears similarities with *trust-region* methods in optimization (see [17] for more details).

Recall first that the family of vectors fields $X = \{X_1, \ldots, X_m\}$ is assumed to be free up to step r. As a consequence the weights $(w_1, \ldots, w_n)$ are equal at every point $a \in \mathcal{V}^c$ to $(\widetilde{w}_1, \ldots, \widetilde{w}_n)$, the free weights of step r. Hence the pseudo-norm $\|\cdot\|_a$ (see Definition 3.13) does not depend on $a \in \mathcal{V}^c$ and will be denoted as $\|\cdot\|_r$.

The parameterized path $t \mapsto \delta_t(x)$ is defined by

$$\delta_t(x) := (t^{w_1} z_1(x), \ldots, t^{w_n} z_n(x)), \quad \text{for } x \in \Omega,$$

where $z := \Phi^X(x^{\text{final}}, \cdot)$. Note that δ_t is the (weighted) *dilatation* in privileged coordinates at x^{final} with parameter t. In particular, $\|z(\delta_t(x))\|_r = |t|\,\|z(x)\|_r$. We also define the function Subgoal as follows.

Subgoal$(\overline{x}, \eta, j)$

1. $t_j := \max(0, 1 - \frac{j\eta}{\|z(\overline{x})\|_r})$;
2. Subgoal$(\overline{x}, \eta, j) := \delta_{t_j}(\overline{x})$

We note that the formula for generating t_j guarantees that

$$\|z(\text{Subgoal}(\overline{x}, \eta, j)) - z(\text{Subgoal}(\overline{x}, \eta, j-1))\|_r \leq \eta,$$

and that $x^d = x^{\text{final}}$ for j large enough.

The global convergence of Algorithm 6 is established in the following theorem. For the sake of clarity, we first assume that the sequences $(x_i)_{i\geq 0}$ and $(x_i^d)_{i\geq 0}$ constructed by Algorithm 6 both stay within $\mathcal{V}^c$. This assumption being of a purely numerical nature, we explain at the end of this section how we can remove it by adding suitable intermediate steps to Algorithm 6.

Theorem 3.16. *Let $\mathcal{V}^c \subset \Omega$ be a connected compact set equal to the closure of its interior. Assume that*

(i) the approximate system system $\mathcal{A}^X$ is provided with a sub-optimal steering law ;

(ii) the LAS method AppSteer *is associated with $\mathcal{A}^X$ and its steering law ;*

Algorithm 6 GlobalFree $(x^{\text{initial}}, x^{\text{final}}, e, \mathcal{V}^c, \text{AppSteer})$

1: $i := 0\,;\ j := 1\,;$
2: $x_i := x^{\text{initial}}\,;\ \overline{x} := x^{\text{initial}}\,;$
3: $\eta := \|z(x^{\text{initial}})\|_r\,;$ {initial choice of the maximum step size;}
4: **while** $\|z(x_i)\|_r > e$ **do**
5: $x^d := \text{Subgoal}\ (\overline{x}, \eta, j)\,;$
6: $x := \text{AppSteer}\ (x_i, x^d)\,;$
7: **if** $\|\Phi^X(x^d, x)\|_r > \frac{1}{2}\|\Phi^X(x^d, x_i)\|_r$ **then** {if the system is not approaching the subgoal,}
8: $\eta := \frac{\eta}{2}\,;$ {reduce the maximum step size,}
9: $\overline{x} := x_i\,;\ j := 1\,;$ {change the path $\delta_{0,t}(\bar{x})$.}
10: **else**
11: $i := i + 1\,;\ j := j + 1\,;$
12: $x_i := x\,;\ x_i^d := x^d\,;$
13: **return** x_i.

Then, $\forall\ (x^{\text{initial}}, x^{\text{final}}) \in \mathcal{V}^c \times \mathcal{V}^c$, Algorithm 6 *terminates in a finite number of steps for any choice of the tolerance* $e > 0$ *provided that the sequences* $(x_i)_{i\geq 0}$ *and* $(x_i^d)_{i\geq 0}$ *both belong to* $\mathcal{V}^c$.

Proof of Theorem 3.16. Note first that, if the conditional statement of Line 7 is not true for every i greater than some i_0, then $x_i^d = x^{\text{final}}$ after a finite number of iterations. In this case, the error $\|z(x_i)\|_r$ is reduced at each iteration and the algorithm stops when it becomes smaller than the given tolerance e. This happens in particular if $d(x_i, x^d) < \varepsilon_{\mathcal{V}^c}$ for all i greater than i_0 because condition (3.12) is verified. Another preliminary remark is that, due to the continuity of the control distance and of the function $\|z(\cdot)\|_r$, there exists $\overline{\eta} > 0$ such that, for every pair $(x_1, x_2) \in \mathcal{V}^c \times \mathcal{V}^c$, one has

$$\|z(x_1) - z(x_2)\|_r < \overline{\eta} \implies d(x_1, x_2) < \frac{\varepsilon_{\mathcal{V}^c}}{2}. \tag{3.36}$$

In the following, we will prove by induction that if, at some step i_0, one has $\eta < \overline{\eta}$, then, for all $i > i_0$, one has

$$d(x_{i-1}, x_i^d) < (1/2 + \cdots + (1/2)^{i-i_0})\varepsilon_{\mathcal{V}^c} < \varepsilon_{\mathcal{V}^c}.$$

We assume without loss of generality that $i_0 = 0$ and $\overline{x} = x_0$. For $i = 1$, by construction, $x^d = \text{Subgoal}(x_0, \eta, 1)$ and

$$\|z(x_0) - z(x^d)\|_r \leq \eta < \overline{\eta}.$$

In view of (3.36), one then has $d(x_0, x^d) < \varepsilon_{\mathcal{V}^c}/2$. In view of (3.12), the conditional statement of Line 7 is not true, therefore $x_1^d = x^d$ and one has

$$d(x_0, x_1^d) < \varepsilon_{\mathcal{V}^c}/2.$$

Assume now that for $i > 1$ one has :

$$d(x_{i-2}, x_{i-1}^d) < (1/2 + \cdots + (1/2)^{i-1})\varepsilon_{\mathcal{V}^c}. \tag{3.37}$$

The subgoal x_{i-1}^d is of the form Subgoal$(\overline{x}, \eta, j)$. Let $x^d = $ Subgoal$(\overline{x}, \eta, j+1)$. One can write :

$$d(x_{i-1}, x^d) \leq d(x_{i-1}, x_{i-1}^d) + d(x_{i-1}^d, x^d).$$

By construction, it is

$$\|z(x_{i-1}^d) - z(x^d)\|_r \leq \eta < \overline{\eta},$$

which implies $d(x_{i-1}^d, x^d) < \varepsilon_{\mathcal{V}^c}/2$. The induction hypothesis (3.37) implies that

$$d(x_{i-1}, x_{i-1}^d) \leq \frac{1}{2} d(x_{i-2}, x_{i-1}^d).$$

Finally, one gets

$$\begin{aligned} d(x_{i-1}, x^d) &\leq \frac{1}{2} d(x_{i-2}, x_{i-1}^d) + d(x_{i-1}^d, x^d) \\ &\leq (1/2 + \cdots + (1/2)^i) \varepsilon_{\mathcal{V}^c}. \end{aligned}$$

In view of (3.12), the conditional statement of Line 7 is not true, and so $x_i^d = x^d$. This ends the induction.

Notice that, at some step i, $\eta \geq \overline{\eta}$, the conditional statement of Line 7 could be false. In this case, η is decreased as in Line 8. The updating law of η guarantees that after a finite number of iterations of Line 8, there holds $\eta < \overline{\eta}$. This ends the proof.

□

When the working space Ω is equal to the whole $\mathbb{R}^n$, the assumption that the sequences $(x_i)_{i\geq 0}$ and $(x_i^d)_{i\geq 0}$ constructed by Algorithm 6 both stay within a compact set $\mathcal{V}^c$ can be removed. This requires a simple modification of Lines 11 and 12 of Algorithm 6.

We choose a real number R close to one, precisely

$$(\frac{1}{2})^{1/(r+1)^2} < R < 1,$$

where r is the maximum value of the degree of nonholonomy of System (4.5) on $\mathcal{V}^c$. For every non-negative integer k, we set $R_k = 1 + R + \cdots + R^k$. The algorithm is modified as follows. Introduce first a new variable k, and add the initialization $k := 0$. Replace then Lines 11 and 12 of Algorithm 6 by the procedure below.

1: **if** $\|z(x)\|_r \geq R_{k+1} \|z(x^{\text{initial}})\|_r$ **then**
2: $\quad \eta := \frac{\eta}{2}$;
3: **else if** $R_k \|z(x^{\text{initial}})\|_r \leq \|z(x)\|_r < R_{k+1} \|z(x^{\text{initial}})\|_r$ **then**
4: $\quad i := i + 1$; $j := j + 1$;
5: $\quad x_i := x$; $x_i^d := x^d$;
6: $\quad \eta := \frac{\eta}{2}$;
7: $\quad k := k + 1$;
8: **else if** $\|z(x)\|_r \leq R_k \|z(x^{\text{initial}})\|_r$ **then**
9: $\quad i := i + 1$; $j := j + 1$;
10: $\quad x_i := x$; $x_i^d := x^d$;

This procedure guarantees that the sequences $(x_i)_{i\geq 0}$ and $(x_i^d)_{i\geq 0}$ of the algorithm both belong to the compact set

$$K = \{x \in \mathbb{R}^n \ : \ \|z(x)\|_r \leq \frac{1}{1-R}\|z(x^{\text{initial}})\|_r\}.$$

Moreover, at each iteration of the algorithm, the new variable k is such that

$$\|z(x_i)\|_r \geq R_k\|z(x^{\text{initial}})\|_r \quad \Rightarrow \quad \eta \leq \frac{\|z(x^{\text{initial}})\|_r}{2^k}.$$

For the sake of clarity, we state here the complete modified algorithm named as Algorithm 7.

Algorithm 7 GlobalFree_Modified $(x^{\text{initial}}, x^{\text{final}}, e, \mathcal{V}^c, \text{AppSteer})$

1: $i := 0$; $j := 1$;
2: $x_i := x^{\text{initial}}$; $\overline{x} := x^{\text{initial}}$;
3: $\eta := \|z(x^{\text{initial}})\|_r$; {initial choice of the maximum step size ;}
4: **while** $\|z(x_i)\|_r > e$ **do**
5: $x^d := \text{Subgoal } (\overline{x}, \eta, j)$;
6: $x := \text{AppSteer } (x_i, x^d)$;
7: **if** $\|\Phi^X(x^d, x)\|_r > \frac{1}{2}\|\Phi^X(x^d, x_i)\|_r$ **then** {if the system is not approaching the subgoal,}
8: $\eta := \frac{\eta}{2}$; {reduce the maximum step size,}
9: $\overline{x} := x_i$; $j := 1$; {change the path $\delta_{0,t}(\bar{x})$.}
10: **else if** $\|z(x)\|_r \geq R_{k+1}\|z(x^{\text{initial}})\|_r$ **then**
11: $\eta := \frac{\eta}{2}$;
12: **else if** $R_k\|z(x^{\text{initial}})\|_r \leq \|z(x)\|_r < R_{k+1}\|z(x^{\text{initial}})\|_r$ **then**
13: $i := i + 1$; $j := j + 1$;
14: $x_i := x$; $x_i^d := x^d$;
15: $\eta := \frac{\eta}{2}$;
16: $k := k + 1$;
17: **else if** $\|z(x)\|_r \leq R_k\|z(x^{\text{initial}})\|_r$ **then**
18: $i := i + 1$; $j := j + 1$;
19: $x_i := x$; $x_i^d := x^d$;
20: **return** x_i.

Proposition 3.17. *Let $\mathcal{V}^c \subset \Omega$ be a connected compact set equal to the closure of its interior. Under the assumptions* (i) *and* (ii) *of Theorem 3.16, $\forall$ $(x^{\text{initial}}, x^{\text{final}}) \in \mathcal{V}^c \times \mathcal{V}^c$,* Algorithm 7 *terminates in a finite number of iterations for any choice of the tolerance $e > 0$.*

Proof of Proposition 3.17. Notice that Lines 17, 18, and 19 in Algorithm 7 are identical to Lines 10, 11, and 12 in Algorithm 6. Therefore, it is enough to show that, after a finite number of iterations, the condition of Line 17 in Algorithm 7 holds true. Another preliminary remark

is that the distance $\|z(x) - z(y)\|_r$ give a rough estimate of the sub-Riemannian distance. Indeed it follows from Theorem 3.4 that, for every pair of close enough points $(x, y) \in \mathcal{V}^c \times \mathcal{V}^c$, one has

$$\frac{1}{C_0}\|z(x) - z(y)\|_r^{r+1} \leq d(x, y) \leq C_0\|z(x) - z(y)\|_r^{1/(r+1)}, \tag{3.38}$$

where C_0 is a positive constant. As a consequence, Eq. (3.36) holds true if $\overline{\eta} \leq (\varepsilon_{\mathcal{V}^c}/(2C_0))^{r+1}$.

Let us choose a positive $\overline{\eta}$ smaller than $(\varepsilon_{\mathcal{V}^c}/(2C_0))^{r+1}$. We next show that if, at some step i_0, $\eta < \overline{\eta}$, then the case of Line 10 and the one of Line 12 occur only in a finite number of iterations. Recall first that, from the proof of Theorem 3.16, one gets, for every $i > i_0$,

$$\|z(x_i^d)\|_r \leq \|z(x_{i_0})\|_r \quad \text{and} \quad d(x_i, x_i^d) \leq \varepsilon_{\mathcal{V}^c}.$$

In view of Eq. (3.38), an obvious adaptation of the latter proof yields, for every $i > i_0$, $d(x_i, x_i^d) \leq 2C_0\eta^{1/(r+1)}$, and thus

$$\|z(x_i) - z(x_i^d)\|_r \leq (2C_0^2)^{1/(r+1)}\eta^{1/(r+1)^2}.$$

Finally one gets

$$\begin{aligned} \|z(x_i)\|_r &\leq \|z(x_i^d)\|_r + \|z(x_i) - z(x_i^d)\|_r \\ &\leq \|z(x_{i_0})\|_r + (2C_0^2)^{1/(r+1)}\eta^{1/(r+1)^2}. \end{aligned} \tag{3.39}$$

On the other hand, there exists an integer k_0 such that $\eta \geq \frac{\|z(x^{\text{initial}})\|_r}{2^{k_0}}$. This implies that $\|z(x_{i_0})\|_r \leq R_{k_0}\|z(x^{\text{initial}})\|_r$. Up to reducing $\overline{\eta}$, and so increasing k_0, assume

$$(2C_0^2)^{1/(r+1)}\left(\frac{\|z(x^{\text{initial}})\|_r}{2^{k_0}}\right)^{1/(r+1)^2} \leq R^{k_0+1}\|z(x^{\text{initial}})\|_r,$$

since one has chosen $R > (\frac{1}{2})^{1/(r+1)^2}$. Using Eq. (3.39), it holds, for every $i \geq i_0$, $\|z(x_i)\|_r \leq R_{k_0}\|z(x^{\text{initial}})\|_r + R^{k_0+1}\|z(x^{\text{initial}})\|_r = R_{k_0+1}\|z(x^{\text{initial}})\|_r$.

Therefore, the case of Line 10 and the one of Line 12 occur in at most $k_0 + 1$ iterations.

Applying again the arguments of the proof of Theorem 3.16, the conclusion follows.

□

Remark 3.23. It is worth pointing out that the additional steps involved in Algorithm 7 are designed to prevent the sequences $(x_i)_{i\geq 0}$ and $(x_i^d)_{i\geq 0}$ from accumulating toward the boundary of the compact $\mathcal{V}^c$. There exist other numerical artifacts of probabilistic nature which solve this problem. One also deduces from the proof of Proposition 3.17 that if the points x^{initial} and x^{final} are *far* enough from the boundary of $\mathcal{V}^c$, the sequences $(x_i)_{i\geq 0}$ and $(x_i^d)_{i\geq}$ will remain in $\mathcal{V}^c$.

3.5 Exact Steering Method for Nilpotent Systems

In this chapter, we devise an exact steering method for general nilpotent systems. Without loss of generality, we assume that the system

$$X = \{X_1, \ldots, X_m\}$$

is nilpotent of step r, free up to step r, and given in the canonical form in coordinates x. Recall that, under this assumption, the dynamics is written as follows

$$\begin{aligned} \dot{x}_i &= u_i, && \text{if } i = 1, \ldots, m; \\ \dot{x}_I &= \tfrac{1}{k!}x_{I_L}\dot{x}_{I_R}, && \text{if } X_I = \mathrm{ad}^k_{X_{I_L}} X_{I_R}, \quad I_L, I_R \in \mathcal{H}^r, \end{aligned} \tag{3.40}$$

where the components of x are numbered by the elements of $\mathcal{H}^r$, i.e., for $I \in \mathcal{H}^r$, the component x_I corresponds to the element X_I. We also assume that we want to steer the System (3.40) from any point $x \in \mathbb{R}^{\widetilde{n}_r}$ to the origin 0 of $\mathbb{R}^{\widetilde{n}_r}$.

Remark 3.24. Note that these two assumptions are not restrictive since, for general nilpotent systems, in order to steer from x^{initial} to x^{final}, it suffices to apply the Desingularization Algorithm at the final point x^{final} (see also Remark 3.14).

This method can also be applied for the construction of a sub-optimal steering law for the approximate system $\mathcal{A}^X$ defined in Section 3.4.1. For practical uses, we require that the inputs give rise to regular trajectories (i.e., at least C^1), which are not too "complex" in the sense that, during the control process, we do not want the system to stop too many times or to make a large number of maneuvers.

Several methods were proposed in the literature for steering nilpotent systems. In [62], the authors make use of piecewise constant controls and obtain smooth controls by imposing some special parameterization (namely by requiring the control system to stop during the control process). In that case, the regularity of the inputs is recovered by using a reparameterization of the time, which cannot prevent in general the occurrence of cusps or corners for the corresponding trajectories. However, regularity of the *trajectories* is generally mandatory for robotic applications. Therefore, the method proposed in [63] is not adapted to such applications. In [62], the proposed inputs are polynomial functions in time, but an algebraic system must be inverted in order to access to these inputs. Moreover, the size and the degree of this algebraic system increase exponentially with respect to the dimension of state space, and there does not exist a general efficient exact method to solve it. Even the existence of solutions is a non trivial issue. Furthermore, the methods [62] and [63] both make use of exponential coordinates which are not explicit and thus require in general numerical integrations of nonlinear differential equations. That prevents the use of these methods in an iterative scheme such as Algorithm 4. Let us also mention the path approximation method by Liu and Sussmann [68], which uses unbounded sequences of sinusoids. Even though this method bears similar theoretical aspects with our method, it is not adapted from a numerical point of view to the motion planning issue since it relies on a limit process of highly oscillating inputs.

3.5.1 Steering by sinusoids

We consider input functions in the form of linear combinations of sinusoids with integer frequencies. In [76], authors used this family of inputs to control the chained-form systems.

We first note that if every component of the input $u = (u_1, \ldots, u_m)$ in Eq. (3.40) is a linear combination of sinusoids with integer frequencies, then the dynamics of every component in Eq. (3.40) is also a linear combination of sinusoids with integer frequencies, which are themselves linear combinations of frequencies involved in the input u. One may therefore expect to move some components during a 2π time-period without modifying others if the frequencies in u are properly chosen. Due to the triangular form of Eq. (3.40), it is reasonable to expect to move the components of x one after another according to the order "$\prec$" induced by the P. Hall basis. In that case, one must ensure that all the components already moved to their preassigned values return to the same values after each $2\pi-$period of control process, while the component under consideration arrives to its preassigned position. However, all the components cannot be moved independently by using sinusoids. For that purpose, we introduce the following notion of *equivalence*.

Définition 3.29 (*Equivalence*)**.** Two elements X_I and X_J in the P. Hall family are said to be *equivalent* if $\Delta_i(X_I) = \Delta_i(X_J)$ for $i = 1, \dots, m$, where we use $\Delta_i(X_I)$ to denote the number of times X_i occurs in X_I. We write $X_I \sim X_J$ if X_I and X_J are equivalent and *equivalence classes* will be denoted by

$$\mathcal{E}_X(\ell_1, \dots, \ell_m) := \{X_I \mid \Delta_i(X_I) = l_i, \text{ for } i = 1, \dots, m\}.$$

We say that the components x_I and x_J are *equivalent* if the corresponding brackets X_I and X_J are equivalent and *equivalent classes for components* are defined as follows,

$$\mathcal{E}_x(\ell_1, \dots, \ell_m) := \{x_I \mid X_I \in \mathcal{E}_X(\ell_1, \dots, \ell_m)\}.$$

Remark 3.25. We will see in the following subsections that the frequencies occurring in the dynamics of x_I only depend on the equivalence class of x_I, and not on the structure of the bracket X_I. Therefore, the equivalent components (in the sense of Definition 3.29) cannot be moved separately by using sinusoids.

Définition 3.30 (*Ordering of equivalence classes*)**.** Let $\mathcal{E}_x(\ell_1, \dots, \ell_m)$ and $\mathcal{E}_x(\tilde{\ell}_1, \dots, \tilde{\ell}_m)$ be two equivalence classes. $\mathcal{E}_x(\ell_1, \dots, \ell_m)$ is said to be smaller than $\mathcal{E}_x(\tilde{\ell}_1, \dots, \tilde{\ell}_m)$ if the smallest element (in the sense of "$\prec$") in $\mathcal{E}_x(\ell_1, \dots, \ell_m)$ is smaller than the one in $\mathcal{E}_x(\tilde{\ell}_1, \dots, \tilde{\ell}_m)$, and we write (by abuse of notation) $\mathcal{E}_x(\ell_1, \dots, \ell_m) \prec \mathcal{E}_x(\tilde{\ell}_1, \dots, \tilde{\ell}_m)$.

Let $\{\mathcal{E}_x^1, \mathcal{E}_x^2, \dots, \mathcal{E}_x^{\widetilde{N}}\}$ be the partition of the set of the components of x induced by Definition 3.29. Assume that, for every pair $(i, j) \in \{1, \dots, \widetilde{N}\}^2$ with $i < j$, one has $\mathcal{E}_x^i \prec \mathcal{E}_x^j$. Our control strategy consists in displacing these equivalence classes one after another according to the ordering "$\prec$" by using sinusoidal inputs. For every $j = 1, \dots, \widetilde{N}$, the key point is to determine how to construct an input u^j defined on $[0, 2\pi]$ such that the two following conditions are verified :

(C1) under the action of u^j, every element of $\mathcal{E}_x^j$ reaches its preassigned value at $t = 2\pi$;

(C2) under the action of u^j, for every $i < j$, every element of $\mathcal{E}_x^i$ returns at $t = 2\pi$ to its value taken at $t = 0$.

Once one knows how to construct an input u^j verifying (C1) and (C2) for every $j = 1, \dots, \widetilde{N}$, it suffices to *concatenate* them to control the complete system.

Définition 3.31 (*Concatenation*)**.** The concatenation of $u^1, \dots, u^{\widetilde{N}}$ is defined on the interval $[0, 2\widetilde{N}\pi]$ by

$$u^1 * \cdots * u^{\widetilde{N}}(t) := u^j(t - 2(j-1)\pi), \tag{3.41}$$

for $t \in [2(j-1)\pi, 2j\pi]$ and $j \in \{1, \dots, \widetilde{N}\}$.

Remark 3.26. As we will show later (see Remark 3.31), for every positive integer k, it is possible to make C^k concatenations such that the inputs are globally of class C^k and the corresponding trajectories are not only piecewise smooth, but also globally of class C^{k+1}.

3.5.2 Choice of frequencies

In this section, we fix an equivalence class $\mathcal{E}_x^j$. We choose frequencies in u^j such that Conditions (C1) and (C2) are verified. For sake of clarity, we first treat the case $m = 2$ in Subsections 3.5.2.1 and 3.5.2.2., and we show, in Subsection 3.5.2.3, how to adapt the method to greater values of m.

3.5.2.1 A simple case : $m = 2$ and Card $(\mathcal{E}_x^j) = 1$

Let x_I be the only element of $\mathcal{E}_x^j$, and X_I the corresponding bracket. Let $m_1 := \Delta_1(X_I)$, and $m_2 := \Delta_2(X_I)$.

Proposition 3.18. *Consider three positive integers ω_1, ω_2, ω_3, and $\varepsilon \in \{0, 1\}$ such that*

$$\begin{cases} \omega_3 = m_1\omega_1 + (m_2 - 1)\omega_2, \\ \varepsilon = m_1 + m_2 - 1 \pmod 2, \end{cases} \tag{3.42}$$

and

$$\omega_2 > (m_1 + m_2)m_1. \tag{3.43}$$

By choosing properly ζ, the control

$$u_1 = \cos\omega_1 t,\ u_2 = \cos\omega_2 t + \zeta\cos(\omega_3 t - \varepsilon\frac{\pi}{2}), \tag{3.44}$$

steers, during $[0, 2\pi]$, the component x_I from any initial value to any preassigned final value without modifying any component x_J, with $J \prec I$. Moreover, $x_I(2\pi) - x_I(0)$ gives rise to a non zero linear function of ζ, where ζ is the coefficient in front of $\cos(\omega_3 t - \varepsilon\frac{\pi}{2})$ in Eq. (3.44).

The key point is to understand the frequencies occurring in the dynamics $\dot{x}_I$.

Lemma 3.19. *For $J \leq I$, the dynamics $\dot{x}_J$ is a linear combination of cosine functions of the form*

$$\cos\{(\ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3)t - (\ell_3\varepsilon + \ell_1 + \ell_2 + \ell_3 - 1)\frac{\pi}{2}\}, \tag{3.45}$$

where $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}$ satisfy $|\ell_1| \leq m_1$, $|\ell_2| + |\ell_3| \leq m_2$.

In particular, the term

$$\cos[(m_1\omega_1 + (m_2 - 1)\omega_2 - \omega_3)t - (-\varepsilon + m_1 + m_2 - 1)\frac{\pi}{2}]$$

occurs in $\dot{x}_I$ with a zero coefficient depending linearly on ζ.

Proof of Lemma 3.19. The proof goes by induction on $|J|$.

- $|J| = 1$, the result is true since $\dot{x}_{I_1} = u_1$ and $\dot{x}_{I_2} = u_2$.
- *Inductive step :*
  Assume that the result holds true for all $\tilde{J}$ such that $|\tilde{J}| < s$. We show that it remains true for J such that $|J| = s$.
  By construction, we have $X_J = \mathrm{ad}^k_{X_{J_1}} X_{J_2}$ with $|J_1| < s$ and $|J_2| < s$. Then,

$$\dot{x}_J = \frac{1}{k!} x^k_{J_1} \dot{x}_{J_2}, \tag{3.46}$$

  $\dot{x}_{J_2}$ is given by the inductive hypothesis and x_{I_1} is obtained by integration of Eq. (3.45). By using product formulas for cosine function, the result still holds true for J of length s. This ends the proof of Lemma 3.19.

□

Proof of Proposition 3.18. First note that integrating between 0 and 2π a function of the form $\cos(\gamma t + \bar{\gamma}\frac{\pi}{2})$ with $(\gamma, \bar{\gamma}) \in \mathbb{N}^2$ almost always gives 0 except for $\gamma = 0$ and $\bar{\gamma} = 0 \pmod 2$ at the same time. Therefore, in order to obtain a non trivial contribution for x_I, $\dot{x}_I$ must contain some cosine functions verifying the following condition

$$\begin{cases} \ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3 = 0, \\ \ell_3\varepsilon + \ell_1 + m_2 + \ell_3 - 1 \equiv 0 \pmod 2, \end{cases} \tag{3.47}$$

and this condition must not be satisfied by $J \prec I$ in order to avoid a change in the component x_J.
Under conditions (3.42) and (3.43), we claim that

(1) $(m_1, m_2 - 1, -1, \varepsilon)$ is the only 4-tuple verifying (3.47) for x_I, and $x_I(2\pi) - x_I(0)$ is a non zero *linear* function of ζ ;

(2) Eq. (3.47) is never satisfied for x_J with $J < I$.

Indeed, consider $(\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3$ verifying $|\ell_1| \leq m_1$, $|\ell_2| + |\ell_3| \leq m_2$. One has

$$\begin{aligned} & \ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3 \\ = \ & \ell_1\omega_1 + \ell_2\omega_2 + \ell_3((m_2 - 1)\omega_2 + m_1\omega_1) \\ = \ & (\ell_3(m_2 - 1) + \ell_2)\omega_2 + (\ell_1 + \ell_3 m_1)\omega_1. \end{aligned} \tag{3.48}$$

Assume that $\omega_2 > (m_1 + m_2)m_1\omega_1$. Then, except for the 4−tuple $(m_1, m_2, m_3, \varepsilon)$ verifying Eq. (3.42), the only possibility to have the right-hand side of Eq. (3.48) equal to 0 is $\ell_1 = \ell_2 = \ell_3 = 0$. In that case,

$$\ell_1 + \ell_2 + \ell_3 \neq 1 \pmod 2.$$

Then, Eq. (3.47) is not satisfied, and **(2)** is proved.

Due to Eq. (3.44), the power of ζ is equal to the number of times ω_3 occurs in the resonance condition (3.42). The latter is clearly equal to 1. Thus, $x_I(2\pi) - x_I(0)$ gives rise to a linear function of ζ. It remains to show that the coefficient in front of ζ is not equal to zero. By Lemma 3.19, one knows that

$$\begin{aligned} \dot{x}_I \ = \ & g_I \cos\{(m_1\omega_1 + m_2\omega_2)t - (m_1 + m_2 - 1)\frac{\pi}{2}\} \\ & + f_I a \cos\{(m_1\omega_1 + (m_2 - 1)\omega_2 - \omega_3)t \\ & - (m_1 + m_2 - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}, \end{aligned} \tag{3.49}$$

where we gathered all other terms into $\mathcal{R}$. Note that the numerical coefficients f_I and g_I depend on the frequencies ω_1, ω_2, and ω_3. The goal is to show that f_I is not equal to zero if we want to move the component x_I, i.e., when $\omega_3 = (m_2 - 1)\omega_2 + m_1\omega_1$. If we consider f_I as a function of ω_1, ω_2, and ω_3, it suffices to show that this function is not identically equal to zero over the hyperplane of $\mathbb{R}^3$ defined by the resonance condition $\omega_3 = (m_2 - 1)\omega_2 + m_1\omega_1$. We assume that the next lemma holds true, and we will provide an argument immediately after finishing the proof of Proposition 3.18.

Lemma 3.20. *For all $J \leq I$, let $m_1^J := \Delta_1(X_J)$ and $m_2^J := \Delta_2(X_J)$. If f_J is the coefficient in front of the term $\cos\{(m_1^J\omega_1 + (m_2^J - 1)\omega_2 - \omega_3)t - (m_1^J + m_2^J - 1 - \varepsilon)\frac{\pi}{2}\}$, and g_J the one in front of the term $\cos\{(m_1^J\omega_1 + m_2^J\omega_2)t - (m_1^J + m_2^J - 1)\frac{\pi}{2}\}$. Then, the quotient $\alpha_J := f_J/g_J$ verifies the following inductive formula.*

- *If $X_J = X_1$, $\alpha_J = 0$; If $X_J = X_2$, $\alpha_J = 1$;*
- *If $X_J = [X_{J_1}, X_{J_2}]$, α_J is defined by*

$$\alpha_J = \frac{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2}{m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3}\alpha_{J_1} + \alpha_{J_2}.$$

where $m_i^{J_1} = \Delta_i(X_{J_1})$ for $i = 1,\ 2$.

Let us take $\omega_3 = -\omega_2$. It results from Lemma 3.20 that, for every $J \leq I$, one has

$$\alpha_J \ = \ \alpha_{J_1} + \alpha_{J_2}, \ \text{ if } X_J = [X_{J_1}, X_{J_2}].$$

Since $\alpha_1 = 0$ and $\alpha_2 = 1$, then, over the hyperplane of $\mathbb{R}^3$ defined by $\omega_3 = -\omega_2$, the function α_J is a strictly positive number independent of ω_1 and ω_2.

Let us show now that $\alpha_J(\omega_1, \omega_2, \omega_3)$ is not identically equal to zero over the hyperplane of $\mathbb{R}^3$ defined by $\omega_3 = m_1\omega_1 + (m_2 - 1)\omega_2$. Let $\hat{\omega}_2 := -m_1\omega_1/m_2$. One has

$$m_1\omega_1 + (m_2 - 1)\hat{\omega}_2 = -\hat{\omega}_2.$$

It implies that

$$\alpha_I(\omega_1, \hat{\omega}_2, m_1\omega_1 + (m_2 - 1)\hat{\omega}_2) = \alpha_I(\omega_1, \hat{\omega}_2, -\hat{\omega}_2).$$

Since the function $\alpha_I(\omega_1, \omega_2, -\omega_2)$ is never equal to zero, and it coincides with the function $\alpha_I(\omega_1, \omega_2, m_1\omega_1 + (m_2 - 1)\omega_2)$ at the point $(\omega_1, \hat{\omega}_2)$, which is not identically equal to zero.

Therefore, $f_I(\omega_1, \omega_2, \omega_3)$ is not identically equal to zero over the hyperplane $\omega_3 = (m_2 - 1)\omega_2 + m_1\omega_1$. Moreover, as it is a non trivial rational function, it eventually vanishes at a finite number of integer points. Then, we obtain a non zero linear function of ζ, and **(1)** is now proved. Proposition 3.18 results from **(1)** and **(2)**.

□

Proof of Lemma 3.20. The proof goes by induction on $|I|$. Since $\dot{x}_1 = u_1$ and $\dot{x}_2 = u_2$, by Eq. (3.44), one has $\alpha_1 = 0$ and $\alpha_2 = 1$.

Assume that $|J| \geq 2$. By construction, one has $X_J = [X_{J_1}, X_{J_2}]$ with $|J_1| \leq |J_2| < |J|$. According to the inductive hypothesis, one has

$$\begin{aligned}
\dot{x}_{J_1} &= g_{J_1}\cos\{(m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2)t - (m_1^{J_1} + m_2^{J_1} - 1)\frac{\pi}{2}\} \\
&\quad + f_{J_1}\cos\{(m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3)t \\
&\quad - (m_1^{J_1} + m_2^{J_1} - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_{J_1}, \\
\dot{x}_{J_2} &= g_{J_2}\cos\{(m_1^{J_2}\omega_1 + m_2^{J_2}\omega_2)t - (m_1^{J_2} + m_2^{J_2} - 1)\frac{\pi}{2}\} \\
&\quad + f_{J_2}\cos\{(m_1^{J_2}\omega_1 + (m_2^{J_2} - 1)\omega_2 - \omega_3)t \\
&\quad - (m_1^{J_2} + m_2^{J_2} - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_{J_2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\dot{x}_J &= \Bigg(\frac{1}{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2} g_{J_1} \cos\{(m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2)t - (m_1^{J_1} + m_2^{J_1})\frac{\pi}{2}\} \\
&\quad + \frac{1}{m_1^{J_1} + (m_2^{J_1} - 1)\omega_2 - \omega_3} f_{J_1} a \cos\{(m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3)t \\
&\quad - (m_1^{J_1} + m_2^{J_2} - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_{J_1}\Bigg) \\
&\quad \Big(g_{J_2} \cos\{(m_1^{J_2}\omega_1 + m_2^{J_2}\omega_2)t - (m_1^{J_2} + m_2^{J_2} - 1)\frac{\pi}{2}\} \\
&\quad + f_{J_2} a \cos\{(m_1^{J_2}\omega_1 + (m_2^{J_2} - 1)\omega_2 - \omega_3)t \\
&\quad - (m_1^{J_2} + m_2^{J_2} - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_{J_2}\Big) \\
&= \frac{1}{2}\frac{g_{J_1} g_{J_2}}{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2} \cos\{(m_1^J\omega_1 + m_2^J\omega_2)t - (m_1^J + m_2^J - 1)\frac{\pi}{2}\} \\
&\quad + \frac{1}{2}\left(\frac{g_{J_1} f_{J_2}}{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2} + \frac{g_{J_2} f_{J_1}}{m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3}\right) \\
&\quad \cos\{(m_1^J\omega_1 + m_2^J\omega_2 - \omega_3)t - (m_1^J + m_2^J - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_J \\
&= g_J \cos\{(m_1^J\omega_1 + m_2^J\omega_2)t - (m_1^J + m_2^J - 1)\frac{\pi}{2}\} \\
&\quad + f_J \cos\{(m_1^J\omega_1 + m_2^J\omega_2 - \omega_3)t - (m_1^J + m_2^J - 1 - \varepsilon)\frac{\pi}{2}\} + \mathcal{R}_J.
\end{aligned}$$

Therefore, one obtains

$$\alpha_J = \frac{m_1^{J_1}\omega_1 + m_2^{J_1}\omega_2}{m_1^{J_1}\omega_1 + (m_2^{J_1} - 1)\omega_2 - \omega_3}\alpha_{J_1} + \alpha_{J_2}.$$

□

3.5.2.2 A more general case : $m = 2$ and Card $(\mathcal{E}_x^j) > 1$

In general, given a pair (m_1, m_2), the equivalence class $\mathcal{E}_x(m_1, m_2)$ contains more than one element. This situation first occurs for Lie brackets of length 5. For instance, given the pair $(3, 2)$, one has both $X_I = [X_2, [X_1, [X_1, [X_1, X_2]]]]$ and $X_J = [[X_1, X_2], [X_1, [X_1, X_2]]]$. By Lemma 3.19, if one chooses a 4-tuple verifying the resonance condition (3.42) for $\dot{x}_I$, the same resonance occurs in $\dot{x}_J$. Such two components cannot be independently steered by using resonance. The idea is to move simultaneously these components. For instance, one can choose (u_1, u_2) as follows :

$$\begin{aligned}
u_1(t) &= \cos\omega_1 t, \\
u_2(t) &= \cos\omega_2 t + a_I \cos\omega_3 t + \cos\omega_4 t + a_J \cos\omega_5 t,
\end{aligned}$$

where $\omega_1 = 1$, ω_2 is chosen according to Eq. (3.43),

$$\omega_3 = (m_2 - 1)\omega_2 + m_1\omega_1,$$

and

$$\omega_5 = (m_2 - 1)\omega_4 + m_1\omega_1,$$

with ω_4 large enough to guarantee Condition (C2). After explicit integration of Eq. (3.40), one obtains

$$\begin{pmatrix} f_I(\omega_1,\omega_2) & f_I(\omega_1,\omega_4) \\ f_J(\omega_1,\omega_2) & f_J(\omega_1,\omega_4) \end{pmatrix} \begin{pmatrix} a_I \\ a_J \end{pmatrix} = A \begin{pmatrix} a_I \\ a_J \end{pmatrix} = \begin{pmatrix} x_I(2\pi) - x_I(0) \\ x_J(2\pi) - x_J(0) \end{pmatrix},$$

where f_I and f_J are two rational functions of frequencies. Thus, the pair (u_1, u_2) controls exactly and simultaneously x_I and x_J, provided that the matrix A is invertible. We generalize this strategy in the following paragraphs. Assume that $\mathcal{E}_x^j(m_1, m_2) = \{x_{I_1}, \ldots, x_{I_N}\}$. The main result is given next.

Proposition 3.21. *Consider*

$$\{\omega_{11}^1, \ldots, \omega_{11}^{m_1}\}, \ldots, \{\omega_{1N}^1, \ldots, \omega_{1N}^{m_1}\},$$
$$\{\omega_{21}^1, \ldots, \omega_{21}^{m_2-1}, \omega_{21}^*\}, \ldots, \{\omega_{2N}^1, \ldots, \omega_{2N}^{m_2-1}, \omega_{2N}^*\}$$

belonging to $\mathbb{N}^{m_1 N} \times \mathbb{N}^{m_2 N}$ *such that*

$$\begin{cases} \forall\ j = 1 \ldots N, \quad \omega_{2j}^* = \displaystyle\sum_{i=1}^{m_1} \omega_{1j}^i + \sum_{i=1}^{m_2-1} \omega_{2j}^i, \\ \varepsilon = m_1 + m_2 - 1 \pmod 2, \end{cases} \tag{3.50}$$

and

$$\forall j = 1 \ldots N-1, \begin{cases} \omega_{11}^1 & \in & \mathbb{N}\ ; & \\ \omega_{1j}^{i+1} & > & m_1 \omega_{1j}^i\ ; & i = 1 \ldots m_1, \\ \omega_{2j}^1 & > & m_1 \omega_{1j}^{m_1}\ ; & \\ \omega_{2j}^i & > & m_2 \omega_{2j}^{i-1} + m_1 \omega_{1j}^{m_1}\ ; & i = 2 \ldots m_2 - 1, \\ \omega_{1j+1}^1 & > & m_2 \omega_{2j}^{m_2-1} + m_1 \omega_{1j}^{m_1}\ . & \end{cases} \tag{3.51}$$

Then, the input $u := (u_1, u_2)$ *defined by*

$$\begin{cases} u_1 & := & \displaystyle\sum_{j=1}^{N} \sum_{i=1}^{m_1} \cos \omega_{1j}^i t, \\ u_2 & := & \displaystyle\sum_{j=1}^{N} \sum_{i=1}^{m_2-1} \cos \omega_{2j}^i t + a_j \cos(\omega_{2j}^* t - \varepsilon \frac{\pi}{2}), \end{cases} \tag{3.52}$$

steers the components $(x_{I_1}, \ldots, x_{I_N})$ *from an arbitrary initial condition*

$$(x_{I_1}(0), \ldots, x_{I_N}(0))$$

to an arbitrary final one

$$(x_{I_1}(2\pi), \ldots, x_{I_N}(2\pi)),$$

without modifying any other component having been previously moved to its final value.

This result generalizes Proposition 3.18. The proof is decomposed in two parts as follows :

Part I : we show that, if (3.51) holds and the control functions are of the form (3.52), then (3.50) is the *only* resonance occurring in $(\dot{x}_{I_1}, \ldots, \dot{x}_{I_N})$;

Part II : as the resonance gives rise to a system of linear equations on $(a_1, \ldots, a_N)$, we recover the invertibility of this system by choosing suitable frequencies in the control function (3.52).

Part I Frequencies and Resonance

Consider inputs of the form (3.52). Generalizing Lemma 3.19, we give a general form of frequencies involved in $\dot{x}_J$.

Lemma 3.22. *The dynamics $\dot{x}_J$ is a linear combination of cosine functions of the form*

$$(\ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^*)t - (\ell_1 + \ell_2 + m_2^* - 1 + \ell_2^* \varepsilon)\frac{\pi}{2}, \tag{3.53}$$

where

$$\ell_1 \cdot \omega_1 = \sum_{j=1}^{N}\sum_{i=1}^{m_1} \ell_{1j}^i \omega_{1j}^i,\ \ell_2 \cdot \omega_2 = \sum_{j=1}^{N}\sum_{i=1}^{m_2-1} \ell_{2j}^i \omega_{2j}^i,\ \ell_2^* \cdot \omega_2^* = \sum_{j=1}^{N} \ell_{2j}^* \omega_{2j}^*,$$
$$\ell_1 = \sum_{j=1}^{N}\sum_{i=1}^{m_1} \ell_{1j}^i,\ \ell_2 = \sum_{j=1}^{N}\sum_{i=1}^{m_2-1} \ell_{2j}^i,\ \ell_2^* = \sum_{j=1}^{N} m_{2j}^*,$$

with $(\ell_{1j}^i, \ell_{2j}^i, \ell_{2j}^*) \in \mathbb{Z}^3$.

Let

$$|\ell_1| = \sum_{j=1}^{N}\sum_{i=1}^{m_1} |\ell_{1j}^i|,\ |\ell_2| = \sum_{j=1}^{N}\sum_{i=1}^{m_2-1} |\ell_{2j}^i|,\ \textit{and}\ \ |\ell_2^*| = \sum_{j=1}^{N} |\ell_{2j}^*|,$$

then, one has $|\ell_1| \le \Delta_1(X_J)$, $|\ell_2| + |\ell_2^*| \le \Delta_2(X_J)$.

Proof of Lemma 3.22. The proof goes by induction on $|J|$.

- $|J| = 1$: the result is true since $\dot{x}_1 = u_1$ and $\dot{x}_2 = u_2$.
- *Inductive step :*
  Assume that the result holds true for all $x_{\tilde{J}}$ such that $1 \le |\tilde{J}| < s$. We show that it remains true for x_J with $|J| = s$. By construction, we have $X_J = \mathrm{ad}^k_{X_{J_1}} X_{J_2}$, and

$$\dot{x}_J = \frac{1}{k!} x_{J_1}^k \dot{x}_{J_2}, \tag{3.54}$$

  with $|J_1| < |J|$, $|J_2| < |J|$, and $k|J_1| + |J_2| = |J|$.
  Then, by the inductive hypothesis, we have

$$\begin{aligned}
\dot{x}_{J_1} &= \mathrm{LinCom}\left\{\cos\{(\ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^*)t \right.\\
&\quad \left. -(\ell_1 + \ell_2 + \ell_2^* - 1 + \ell_2^* \varepsilon)\frac{\pi}{2}\}\right\},
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
\dot{x}_{J_2} &= \mathrm{LinCom}\left\{\cos\{(\tilde{\ell}_1 \cdot \omega_1 + \tilde{\ell}_2 \cdot \omega_2 + \tilde{\ell}_2^* \cdot \omega_2^*)t \right.\\
&\quad \left. -(\tilde{\ell}_1 + \tilde{\ell}_2 + \tilde{\ell}_2^* - 1 + \tilde{\ell}_2^* \varepsilon)\frac{\pi}{2}\}\right\},
\end{aligned} \tag{3.56}$$

  where LinCom$\{\cdot\}$ stands "linear combination".
  Eq. (3.55) implies that

$$\begin{aligned}
x_{J_1} &= \mathrm{LinCom}\left\{\cos\{(\ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^*)t \right.\\
&\quad \left. -(\ell_1 + \ell_2 + \ell_2^* - 1 + \ell_2^* \varepsilon)\frac{\pi}{2} - \frac{\pi}{2}\}\right\}\\
&= \mathrm{LinCom}\left\{\cos\{(\ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell_2^* \cdot \omega_2^*)t \right.\\
&\quad \left. - (\ell_1 + \ell_2 + \ell_2^* + \ell_2^* \varepsilon)\frac{\pi}{2}\}\right\}.
\end{aligned} \tag{3.57}$$

For notational ease, we will only write down the case

$$\dot{x}_J = x_{J_1}\dot{x}_{J_2}.$$

Using product formulas for cosine function, one has

$$\begin{aligned}\dot{x}_J &= \mathrm{LinCom}\Big\{\cos\{[(\ell_1 \pm \tilde{\ell}_1)\cdot\omega_1 + (\ell_2 \pm \tilde{\ell}_2)\cdot\omega_2 + (\ell_2^* \pm \tilde{\ell}_2^*)\cdot\omega_2^*]t \\ &\quad -[(\ell_1 \pm \tilde{\ell}_1) + (\ell_2 \pm \tilde{\ell}_2) + (\ell_2^* \pm \tilde{\ell}_2^*) - 1 + (\ell_2^* \pm \tilde{\ell}_2^*)\varepsilon]\frac{\pi}{2}\}\Big\}. \qquad (3.58)\end{aligned}$$

Moreover, according to the inductive hypothesis, one has

$$|\ell_1| \le \Delta_1(X_{J_1}),\ |\ell_2| + |\ell_2^*| \le \Delta_2(X_{J_1}),$$

and

$$|\tilde{\ell}_1| \le \Delta_1(X_{J_2}),\ |\tilde{\ell}_2| + |\tilde{\ell}_2^*| \le \Delta_2(X_{J_2}).$$

Then, one gets

$$|\tilde{\ell}_1 \pm \tilde{\ell}_1| \le \Delta_1(X_J),\ \text{and}\ |\tilde{\ell}_2 \pm \tilde{\ell}_2| + |\ell_2^* \pm \tilde{\ell}_2^*| \le \Delta_2(X_J).$$

This concludes the proof of Lemma 3.22. □

By Lemma 3.22, one gets a non trivial contribution for x_J if the resonance condition

$$\begin{cases} \ell_1\cdot\omega_1 + \ell_2\cdot\omega_2 + \ell_2^*\cdot\omega_2^* = 0, \\ \ell_2^*\varepsilon + \ell_1 + \ell_2 + \ell_2^* - 1 \equiv 0 \pmod 2, \end{cases} \qquad (3.59)$$

is verified by the frequencies of some cosine functions involved in $\dot{x}_J$.

Lemma 3.23. *Under conditions* (3.50) *and* (3.51) *in Proposition 3.21, one gets a non trivial contribution on x_{I_j} depending linearly on a_j for all $j = 1\ldots, N$.*

Proof of Lemma 3.23. It is clear that the resonance condition (3.59) holds for

$$\{\omega_{11}^1,\ldots,\omega_{11}^{m_1}\},\ldots,\{\omega_{1N}^1,\ldots,\omega_{1N}^{m_1}\},$$

$$\{\omega_{21}^1,\ldots,\omega_{21}^{m_2-1},\omega_{21}^*\},\ldots,\{\omega_{2N}^1,\ldots,\omega_{2N}^{m_2-1},\omega_{2N}^*\},$$

and $\varepsilon \in \{0,1\}$ verifying (3.50). We show that it is the only resonance occurring in $\dot{x}_{I_j}$. Indeed, by Lemma 3.22, the integer part of frequencies in $\dot{x}_{I_j}$ is in the following form

$$\begin{aligned}&\ell_1\cdot\omega_1 + \ell_2\cdot\omega_2 + \ell_2^*\cdot\omega_2^* \\ =\ &\sum_{j=1}^N\sum_{i=1}^{m_1}\ell_{1j}^i\omega_{1j}^i + \sum_{j=1}^N\sum_{i=1}^{m_2-1}\ell_{2j}^i\omega_{2j}^i + \sum_{j=1}^N\ell_{2j}^*\omega_{2j}^* \\ =\ &\sum_{j=1}^N\sum_{i=1}^{m_1}\ell_{1j}^i\omega_{1j}^i + \sum_{j=1}^N\sum_{i=1}^{m_2-1}\ell_{2j}^i\omega_{2j}^i \\ &+\sum_{j=1}^N\ell_{2j}^*\left(\sum_{i=1}^{m_1}\omega_{1j}^i + \sum_{i=1}^{m_2-1}\omega_{2j}^i\right) \\ =\ &\sum_{j=1}^N\sum_{i=1}^{m_1}(\ell_{1j}^i + \ell_{2j}^*)\omega_{1j}^i + \sum_{j=1}^N\sum_{i=1}^{m_2-1}(\ell_{2j}^i + \ell_{2j}^*)\omega_{2j}^i. \qquad (3.60)\end{aligned}$$

By Condition (3.51), Eq. (3.60) is equal to zero if and only if

$$\begin{aligned} \ell^i_{1j} + \ell^*_{2j} &= 0, \quad \text{for } i = 1, \dots, m_1, \\ \ell^i_{2j} + \ell^*_{2j} &= 0, \quad \text{for } i = 1, \dots, m_2 - 1. \end{aligned}$$

Then, one has

$$\begin{aligned} |\ell_1| &= \sum_{j=1}^{N} \sum_{i=1}^{m_1} |\ell^i_{1j}| = \sum_{j=1}^{N} \sum_{i=1}^{m_1} |\ell^*_{2j}| = m_1 \sum_{j=1}^{N} |\ell^*_{2j}|, \\ |\ell_2| &= \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} |\ell^i_{2j}| = \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} |\ell^*_{2j}| = (m_2 - 1) \sum_{j=1}^{N} |\ell^*_{2j}|. \end{aligned}$$

However, by Lemma 3.22, one knows that $|\ell_1| \leq m_1$ and $|\ell_2| + |\ell^*_2| \leq m_2$. Then, one necessarily has $m^*_{2j} = 0$ for all $j = 1, \dots, N$. In that case, one obtains

$$\ell^*_2 \varepsilon + \ell_1 + \ell_2 + \ell^*_2 - 1 = -1 \neq 0 \pmod 2.$$

In conclusion, the resonance condition (3.59) does not hold for any 4−tuple $(\ell_1, \ell_2, \ell^*_2, \varepsilon)$ different from $(m_1, m_2 - 1, -1, m_1 + m_2 - 1 \pmod 2)$.

By Eq. (3.52), the power of a_j is equal to the number of times ω^*_{2j} occurs in the resonance condition (3.42). Since the latter is equal to 1, we obtain a linear function of a_j. This ends the proof of Lemma 3.23.

□

Lemma 3.24. *If $x_J \in \mathcal{E}^i_x$ and $i < j$, then $x_J(2\pi) - x_J(2\pi) = 0$.*

Proof of Lemma 3.24. We first note that Eq. (3.60) still holds true. Recall its expression here.

$$\begin{aligned} & \ell_1 \cdot \omega_1 + \ell_2 \cdot \omega_2 + \ell^*_2 \cdot \omega^*_2 \\ = & \sum_{j=1}^{N} \sum_{i=1}^{m_1} (\ell^i_{1j} + \ell^*_{2j}) \cdot \omega^i_{1j} + \sum_{j=1}^{N} \sum_{i=1}^{m_2-1} (\ell^i_{2j} + \ell^*_{2j}) \cdot \omega^i_{2j} \end{aligned} \tag{3.61}$$

By condition (3.51) in Proposition 3.21, Eq. (3.61) is equal to zero if and only if $\ell^i_{1j} + \ell^*_{2j} = 0$ for $i = 1, \dots, m_1$, $j = 1, \dots, N$ and $\ell^i_{2j} + \ell^*_{2j} = 0$ for $i = 1, \dots, m_2 - 1$, $j = 1, \dots, N$. In that case, one has

$$|\ell_1| = m_1 \sum_{j=1}^{N} |\ell^*_{2j}|, \qquad |\ell_2| + |\ell^*_2| = m_2 \sum_{j=1}^{N} |\ell^*_{2j}|.$$

One also knows that

$$|\ell_1| \leq \Delta_1(X_J), \quad |\ell_2| + |\ell^*_2| \leq \Delta_2(X_J),$$

with $\Delta_1(X_J) < m_1$ or $\Delta_2(X_J) < m_2$. Therefore, one has $\ell^*_{2j} = 0$ for all $j = 1, \dots, N$. This implies that

$$\ell^*_2 \varepsilon + \ell_1 + \ell_2 + \ell^*_2 - 1 = -1 \neq 0 \pmod 2.$$

In conclusion, the resonance condition (3.59) does not hold true. This ends the proof of Lemma 3.24.

□

Part II Invertibility

As a consequence of Lemma 3.23, one has

$$\begin{aligned}\begin{pmatrix} x_{I_1}(2\pi)-x_{I_1}(0) \\ \vdots \\ x_{I_N}(2\pi)-x_{I_N}(0)\end{pmatrix} &= A(\omega_{11}^1,\dots,\omega_{2N}^{m_2-1},\omega_{2N}^*)\begin{pmatrix} a_1 \\ \vdots \\ a_N\end{pmatrix} \\ = \begin{pmatrix} f_{I_1}^X(\omega_{11}^1,\dots,\omega_{21}^*), & \dots & ,f_{I_1}^X(\omega_{1N}^1,\dots,\omega_{2N}^*) \\ \vdots & \ddots & \vdots \\ f_{I_N}^X(\omega_{11}^1,\dots,\omega_{21}^*), & \dots & ,f_{I_N}^X(\omega_{1N}^1,\dots,\omega_{2N}^*)\end{pmatrix}&\begin{pmatrix} a_1 \\ \vdots \\ a_N\end{pmatrix},\end{aligned} \tag{3.62}$$

where $f_{I_j}^X : \mathbb{R}^m \to \mathbb{R}$ are rational functions of frequencies, and every ω_{2j}^* verifies Eq. (3.50) for $j = 1,\dots,N$.

Définition 3.32 (*Control matrix and control vector*)**.** The matrix A and the vector $(a_1,\dots,a_N)$ occurring in Eq. (3.62) are called respectively *control matrix* and *control vector* associated with the equivalence class $\mathcal{E}_x^j$.

We show in the sequel that it is possible to choose integer frequencies

$$\{\omega_{11}^1,\dots,\omega_{11}^{m_1}\},\dots,\{\omega_{1N}^1,\dots,\omega_{1N}^{m_1}\},$$
$$\{\omega_{21}^1,\dots,\omega_{21}^{m_2-1},\omega_{21}^*\},\dots,\{\omega_{2N}^1,\dots,\omega_{2N}^{m_2-1},\omega_{2N}^*\},$$

so that the invertibility of the control matrix A is guaranteed, as well as the non-resonance of every component x_J belonging to a class smaller than $\mathcal{E}_x^j$.

For $j = 1,\dots,N$, we use P_j to denote the hyperplane in $\mathbb{R}^M$ with $M := m_1 + m_2$ defined by Eq. (3.50), for which we recall the expression next,

$$\omega_{2j}^* = \sum_{i=1}^{m_1}\omega_{1j}^i + \sum_{i=1}^{m_2-1}\omega_{2j}^i.$$

We start by showing that the function $\det A(\omega_{11}^1,\dots,\omega_{2N}^*)$ is not identically equal to zero on $\cap_{j=1}^N P_j$. This is a consequence of the following lemma.

Lemma 3.25. *The family of functions*

$$\{f_{I_1}^X(\omega_1^1,\dots,\omega_2^{m_2-1},\omega_2^*),\dots,f_{I_N}^X(\omega_1^1,\dots,\omega_2^{m_2-1},\omega_2^*)\}$$

is linearly independent on the hyperplane P in $\mathbb{R}^M$ defined by the equation

$$\omega_2^* = \sum_{i=1}^{m_1}\omega_1^i + \sum_{i=1}^{m_2-1}\omega_2^i.$$

Proof of Lemma 3.25. The first part of the argument consists in considering a family of M indeterminates $Y = \{Y_1,\dots,Y_M\}$ and the associated control system

$$\dot{y} = \sum_{i=1}^{M} v_i Y_i. \tag{3.63}$$

Let H_Y be a *P. Hall family* over Y. Consider the elements

$$\{Y_{J_1},\dots,Y_{J_{\tilde{N}}}\}$$

in H_Y of length M such that $\Delta_i(Y_{J_j}) = 1$, for $i = 1 \dots M$, and $j = 1, \dots, \tilde{N}$, and the corresponding components $\{y_{J_1}, \dots, y_{J_{\tilde{N}}}\}$ in exponential coordinates.

If we apply one control of the form $\{v_i = \cos \nu_i t\}_{i=1\dots M}$, with $\nu_m = \sum_{i=1}^{m-1} \nu_i$, to System (3.63), then, by explicit integration, there exists, for each component y_{J_j}, a fractional function $f^Y_{J_j} : \mathbb{R}^m \to \mathbb{R}$ such that

$$y_{J_j}(2\pi) - y_{J_j}(0) = f^Y_{J_j}(\nu_1, \dots, \nu_M), \quad \text{for } \nu_M = \sum_{i=1}^{M-1} \nu_i. \tag{3.64}$$

Claim 11. *The family of functions $\{f^Y_{J_1}, \dots, f^Y_{I_{\tilde{N}}}\}$ is linearly independent on the hyperplane in $\mathbb{R}^M$ defined by $\nu_M = \sum_{i=1}^{M-1} \nu_i$.*

Proof of Claim 11. We first define $\tilde{f}^Y_{J_j}$ by

$$\tilde{f}^Y_{J_j}(\nu_1, \dots, \nu_M) = f^Y_{J_j}(\nu_1, \dots, -\nu_M). \tag{3.65}$$

Then, it is easy to see that $\tilde{f}^Y_{J_j}$ verifies the following inductive formula :

1. for $J = i = 1 \dots M$, $\tilde{f}^Y_J(\nu_i) = \frac{1}{\nu_i}$;
2. for $|J| > 1$, $Y_J = [Y_{J_1}, Y_{J_2}]$, there exists an injective function

$$\sigma_J : \{1, \dots, m^J\} \to \{1, \dots, M\}$$

such that

$$\begin{aligned} & \tilde{f}^Y_J(\nu_{\sigma_J(1)}, \dots, \nu_{\sigma_J(m^J)}) \\ = \; & \frac{\tilde{f}^Y_{J_1}(\nu_{\sigma_J(1)}, \dots, \nu_{\sigma_J(m^{J_1})})}{\sum_{i=1}^{m^{J_1}} \nu_{\sigma_J(i)}} \tilde{f}^Y_{J_2}(\nu_{\sigma_J(m^{J_1}+1)}, \dots, \nu_{\sigma_J(m^J)}), \end{aligned} \tag{3.66}$$

where $m^J := \Delta(Y_J)$, $m^{J_1} := \Delta(Y_{J_1})$, and $m^{J_2} := \Delta(Y_{J_2})$.

We note that the family of rational functions $\tilde{f}^Y_J$ is well defined for all the Lie brackets Y_J such that $\Delta_i(Y_J) \leq 1$, $i = 1, \dots, M$. The algebraic construction could be extended to all the Lie brackets, but it is not necessary for our purpose. We also note that Claim 11 is equivalent to the fact that the family of rational functions

$$\{\tilde{f}^Y_{J_j}(\nu_1, \dots, \nu_M)\}_{j=1,\dots,\tilde{N}}$$

is linearly independent over the hyperplane $\sum_{i=1}^{M} \nu_i = 0$.

Recall that every element Y_{J_j} in the family $\{Y_{J_1}, \dots, Y_{J_{\tilde{N}}}\}$ writes uniquely as

$$Y_{J_j} = [Y_{J_{j_1}}, Y_{J_{j_2}}]. \tag{3.67}$$

Définition 3.33 (*Left and right factors*)**.** For $J \in \{J_1, \dots, J_{\tilde{N}}\}$, the *left factor* $L(J)$ and the *right factor* $R(J)$ of J are defined in such a way that $Y_J = [Y_{L(J)}, Y_{R(J)}]$.

Let L^* be defined by

$$L^* := \max_{j=1,\dots,\tilde{N}} \{L(J_j)\}. \tag{3.68}$$

The integer L^* is well defined since a P. Hall family is totally ordered. Thus, there exists $J^* \in \{J_1, \dots, J_{\tilde{N}}\}$ such that $L^* = L(J^*)$. Then, define $R^* := R(J^*)$ and set $m^* = |L^*|$. Let

$$\Lambda = \Lambda_L \cup \Lambda_R \text{ and } \bar{\Lambda} = \{1, \dots, \tilde{N}\} \setminus \Lambda,$$

with Λ_L and Λ_R defined by

$$\Lambda_L := \{j \in \{1,\dots,\tilde{N}\}, \text{ such that } Y_{L(J_j)} \sim Y_L\}, \tag{3.69}$$

$$\Lambda_R := \{j \in \{1,\dots,\tilde{N}\}, \text{ such that } Y_{L(J_j)} \sim Y_R\}. \tag{3.70}$$

Then, for all $j \in \Lambda$, there exists an injection function

$$\sigma_j : \{1,\dots,M\} \to \{1,\dots,M\}$$

such that one has

$$\tilde{f}^Y_{J_j}(\nu_1,\dots,\nu_M) \tag{3.71}$$

$$= \frac{\tilde{f}^Y_{L(J_j)}(\nu_{\sigma_j(1)},\dots,\nu_{\sigma_j(m^*)})}{\sum_{i=1}^{m^*}\nu_{\sigma_j(i)}}\tilde{f}^Y_{R(J_j)}(\nu_{\sigma_j(m^*+1)},\dots,\nu_{\sigma_j(M)}), \quad \text{if } j \in \Lambda_L,$$

$$\tilde{f}^Y_{J_j}(\nu_1,\dots,\nu_M) \tag{3.72}$$

$$= \frac{\tilde{f}^Y_{L(J_j)}(\nu_{\sigma_j(m^*+1)},\dots,\nu_{\sigma_j(M)})}{\sum_{i=m^*+1}^{M}\nu_{\sigma_j(i)}}\tilde{f}^Y_{R(J_j)}(\nu_{\sigma_j(1)},\dots,\nu_{\sigma(m^*)}), \quad \text{if } j \in \Lambda_R.$$

Note that, for all j_1 and j_2 in Λ_L, one has

$$\{\nu_{\sigma_{j_1}(1)},\dots,\nu_{\sigma_{j_1}(m^*)}\} = \{\nu_{\sigma_{j_2}(1)},\dots,\nu_{\sigma_{j_2}(m^*)}\}.$$

Denote by Ξ_L the set of variables involved in $\tilde{f}^Y_{L(J_j)}$ with $j \in \Lambda_L$. A similar property holds for Λ_R. For all j_1 and j_2 in Λ_R, one has

$$\{\nu_{\sigma_{j_1}(m^*+1)},\dots,\nu_{\sigma_{j_1}(M)}\} = \{\nu_{\sigma_{j_2}(m^*+1)},\dots,\nu_{\sigma_{j_2}(M)}\}.$$

Denote by Ξ_R the set of all variables occurring in $\tilde{f}^Y_{L(J_j)}$ with $j \in \Lambda_R$. Then one has

$$\Xi_L \cup \Xi_R = \{\nu_1,\dots,\nu_M\}.$$

By abuse of notation, we re-write Eqs. (3.71) and (3.72) in the following form :

$$\tilde{f}^Y_{J_j}(\nu_1,\dots,\nu_M) = \frac{\tilde{f}^Y_{L(J_j)}(\Xi_L)}{\sum_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k}\tilde{f}^Y_{R(J_j)}(\Xi_R), \quad \text{if } j \in \Lambda_L; \tag{3.73}$$

$$\tilde{f}^Y_{J_j}(\nu_1,\dots,\nu_M) = \frac{\tilde{f}^Y_{L(J_j)}(\Xi_R)}{\sum_{\tilde{\nu}_k\in\Xi_R}\tilde{\nu}_k}\tilde{f}^Y_{R(J_j)}(\Xi_L), \quad \text{if } j \in \Lambda_R. \tag{3.74}$$

Moreover, by the resonance condition $\sum_{i=1}^{M}\nu_i = 0$, Eq. (3.74) becomes

$$\tilde{f}^Y_{J_j}(\nu_1,\dots,\nu_M) = \frac{\tilde{f}^Y_{L(J_j)}(\Xi_R)}{-\sum_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k}\tilde{f}^Y_{R(J_j)}(\Xi_L), \quad \text{if } j \in \Lambda_R. \tag{3.75}$$

We now prove that the family of rational functions

$$\{\tilde{f}^Y_{J_j}(\nu_1,\dots,\nu_M)\}_{j=1,\dots,\tilde{N}}$$

is linearly independent over the hyperplane $\sum_{i=1}^{M}\nu_i = 0$. The proof goes by induction over the length of the Lie brackets under consideration. For the brackets of length 1, the result is

obviously true. Assume that the result holds for all brackets of length smaller than $M-1$, $M \geq 2$.

Assume that there exist $\ell_j \in \mathbb{R}^{\tilde{N}}$ such that

$$\sum_{j=1}^{\tilde{N}} \ell_j \tilde{f}^Y_{J_j}(\nu_1, \ldots, \nu_M) = 0, \quad \text{with } \sum_{i=1}^{M} \nu_i = 0. \tag{3.76}$$

One has

$$\begin{aligned} & \sum_{j=1}^{\tilde{N}} \ell_j \tilde{f}^Y_{J_j}(\nu_1, \ldots, \nu_M) \qquad (3.77)\\ = & \sum_{j\in\Lambda} \ell_j \tilde{f}^Y_{J_j}(\nu_1, \ldots, \nu_M) + \sum_{j\in\bar{\Lambda}} \ell_j \tilde{f}^Y_{J_j}(\nu_1, \ldots, \nu_M) \\ = & \sum_{j\in\Lambda_L} \ell_j \frac{\tilde{f}^Y_{L(J_j)}(\Xi_L)}{\sum_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k} \tilde{f}^Y_{R(J_j)}(\Xi_R) - \sum_{j\in\Lambda_R} \ell_j \frac{\tilde{f}^Y_{L(J_j)}(\Xi_R)}{\sum_{\tilde{\nu}_k\in\Xi_L}\tilde{\nu}_k} \tilde{f}^Y_{R(J_j)}(\Xi_L) \\ & + \sum_{j\in\bar{\Lambda}} \ell_j \tilde{f}^Y_{J_j}(\nu_1, \ldots, \nu_M) \\ = & \ 0. \end{aligned}$$

Multiplying Eq. (3.77) by the factor $\sum_{\tilde{\nu}_k\in\Xi_L} \tilde{\nu}_k$, one gets

$$\begin{aligned} & \sum_{j\in\Lambda_L} \ell_j \tilde{f}^Y_{L(J_j)}(\Xi_L) \tilde{f}^Y_{R(J_j)}(\Xi_R) - \sum_{j\in\Lambda_R} \ell_j \tilde{f}^Y_{L(J_j)}(\Xi_R) \tilde{f}^Y_{R(J_j)}(\Xi_L) \\ + & \ (\sum_{\tilde{\nu}_k\in\Xi_L} \tilde{\nu}_k) \sum_{j\in\bar{\Lambda}} \ell_j \tilde{f}^Y_{J_j}(\nu_1, \ldots, \nu_M) = 0. \end{aligned} \tag{3.78}$$

Since L^* is the maximal element among the left factors of Lie brackets of length M, the fraction $\tilde{f}^Y_{J_j}$ does not contain the factor $\sum_{\tilde{\nu}_k\in\Xi_L} \tilde{\nu}_k$ for all $j \in \bar{\Lambda}$. Therefore, on the hyperplane of $\mathbb{R}^{m^*}$ defined by $\sum_{\tilde{\nu}_k\in\Xi_L} \tilde{\nu}_k = 0$, one has

$$\sum_{j\in\Lambda_L} \ell_j \tilde{f}^Y_{L(J_j)}(\Xi_L) \tilde{f}^Y_{R(J_j)}(\Xi_R) - \sum_{j\in\Lambda_R} \ell_j \tilde{f}^Y_{L(J_j)}(\Xi_R) \tilde{f}^Y_{R(J_j)}(\Xi_L) = 0. \tag{3.79}$$

Fixing variables belonging to Ξ_R, Eq. (3.79) is a linear combination of elements of the family

$$\{\tilde{f}^Y_{L(J_j)}(\Xi_L)\}_{j\in\Lambda_L} \cup \{\tilde{f}^Y_{R(J_j)}(\Xi_L)\}_{j\in\Lambda_R}$$

associated with elements of length m^* in the P. Hall family. By the inductive hypothesis, this family is linearly independent over the hyperplane of $\mathbb{R}^{m^*}$ defined by $\sum_{\tilde{\nu}_k\in\Xi_L} \tilde{\nu}_k = 0$. We therefore obtain that

$$\ell_j \tilde{f}^Y_{R(J_j)}(\Xi_R) = 0, \qquad \text{for all } j \in \Lambda_L, \tag{3.80}$$

$$\ell_j \tilde{f}^Y_{L(J_j)}(\Xi_R) = 0, \qquad \text{for all } j \in \Lambda_R. \tag{3.81}$$

Since Eqs. (3.80) and (3.81) hold true over the whole hyperplane of $\mathbb{R}^{M-m^*}$ defined by $\sum_{\tilde{\nu}_k \in \Xi_R} \tilde{\nu}_k = 0$, one has $\ell_j = 0$ for every $j \in \Lambda$. Therefore, Eq. (3.77) becomes

$$\sum_{j \in \bar{\Lambda}} \ell_j \tilde{f}^Y_{J_j}(\nu_1, \dots, \nu_M) = 0. \tag{3.82}$$

Consider now the maximum left factor for $j \in \bar{\Lambda}$ and iterate the same reasoning used for Eq. (3.76). We deduce that $\ell_j = 0$ for every $j \in \bar{\Lambda}$. Therefore, the family $\{\tilde{f}^Y_{J_j}(\nu_1, \dots, \nu_M)\}_{j=1,\dots,\tilde{N}}$ is linearly independent over the hyperplane $\sum_{i=1}^M \nu_i = 0$ and this concludes the proof of Claim 11.

□

We are now in a position to proceed with the argument of Lemma 3.25. Let X_I be an element of $\mathcal{E}_X(m_1, m_2)$, $M := m_1 + m_2$ and $N :=$Card $\mathcal{E}_X(m_1, m_2)$. Consider also another family of M indeterminates $Y = \{Y_1, \dots, Y_M\}$ and let H_Y be the *P. Hall family* over Y. Finally, consider all the elements of the class $\mathcal{E}_Y(1, \dots, 1) = \{Y_{J_1}, \dots, Y_{J_{\tilde{N}}}\}$ in H_Y.

Let Π be the algebra homomorphism from $L(Y)$ to $L(X)$ defined by

$$\Pi(Y_i) = X_1, \quad \text{for} \quad i = 1, \dots, m_1, \tag{3.83}$$
$$\Pi(Y_i) = X_2, \quad \text{for} \quad i = m_1 + 1, \dots, M. \tag{3.84}$$

Note that the map Π is surjective from $\mathcal{E}_Y$ onto $\mathcal{E}_X$. Consider the following vector fields

$$V_Y = \{v_1 Y_1 + \dots + v_M Y_M\},$$

where

$$v_i = \cos \omega_i t, \text{ for } i = 1 \dots M-1, \text{ and } v_M = \cos(\omega_M t + \varepsilon \frac{\pi}{2}), \tag{3.85}$$

with $\omega_M = \sum_{i=1}^{M-1} \omega_i$, and ω_i verifying the non-resonance conditions.

Then, the non-autonomous flow of V_Y between 0 and 2π is given by

$$\overrightarrow{\exp}(V_Y)(0, 2\pi) = e^{f^Y_{J_1} Y_{J_1}} \circ \dots \circ e^{f^Y_{J_{\tilde{N}}} Y_{J_{\tilde{N}}}} \circ \prod_{J > J_1} e^{f^Y_J Y_J}. \tag{3.86}$$

Let us now apply Π to V_Y, we get

$$\Pi(V_Y) := V^X = \{v_1 \Pi(Y_1) + \dots + v_m \Pi(Y_m)\} = \{u_1 X_1 + u_2 X_2\}, \tag{3.87}$$

where

$$u_1 = \sum_{i=1}^{m_1} v_i = \sum_{i=1}^{m_1} \cos \omega_i t, \tag{3.88}$$

$$u_2 = \sum_{i=m_1+1}^{m} v_i = \sum_{i=m_1+1}^{m-1} \cos \omega_i t + \cos(\omega_m t + \varepsilon \frac{\pi}{2}). \tag{3.89}$$

Then, the non-autonomous flow of V_X between 0 and 2π is given by

$$\begin{aligned} \overrightarrow{\exp}(V_X)(0, 2\pi) &= e^{f^Y_{J_1} \Pi(Y_{J_1})} \circ \dots \circ e^{f^Y_{J_{\tilde{N}}} \Pi(Y_{J_{\tilde{N}}})} \circ \prod_{J > J_1} e^{f^Y_J \Pi(Y_J)} \\ &= e^{\sum_{j=1}^{\tilde{N}} f^Y_{J_j} \Pi(Y_{J_j})} \circ \prod_{J > J_1} e^{\bar{f}^Y_J \Pi(Y_J)}. \end{aligned} \tag{3.90}$$

We also know that

$$\begin{aligned}\overrightarrow{\exp}(V_X)(0,2\pi) &= e^{f^X_{I_1}X_{I_1}} \circ \cdots \circ e^{f^X_{I_N}X_{I_N}} \circ \prod_{I>I_1} e^{f^X_I X_I} \\ &= e^{\sum_{j=1}^N f^X_{I_j}X_{I_j}} \circ \prod_{I>I_1} e^{\bar{f}^X_I X_I}. \end{aligned} \tag{3.91}$$

Recall that Π is surjective from $\mathcal{E}_Y(1,\dots,1)$ onto $\mathcal{E}_X(m_1,m_2)$. Therefore, by identifying Eqs. (3.90) and (3.91), we obtain that, for all $j=1,\dots,N$, $f^X_{I_j}$ is a linear combination of $f^Y_{J_i}$ with $i=1,\dots,\tilde{N}$, i.e.,

$$f^X_{I_j} = \sum_{i=1}^{\tilde{N}} \alpha_i^j f^Y_{J_i}. \tag{3.92}$$

Since the family $(f^Y_{J_i})_{i=1,\dots,\tilde{N}}$ is linearly independent and the matrix $A := (\alpha_i^j)_{i=1,\dots,N;j=1,\dots,\tilde{N}}$ is surjective, we conclude that the family $(f^X_{I_j})_{j=1,\dots,N}$ is also linearly independent. This ends the proof of Lemma 3.25.

□

A consequence of Lemma 3.25 is the following.

Corollary 3.26. *With the above notations, the function* det$A$ *is not identically equal to zero on* $\cap_{j=1}^N P_j$.

Proof of Corollary 3.26. For $j=1,\dots,N$, we define the vector L_j by

$$L_j = \left(f^X_{I_j}(\omega^1_{11},\dots,\omega^{m_2-1}_{21},\omega^*_{21}),\cdots,f^X_{I_j}(\omega^1_{1N},\dots,\omega^{m_2-1}_{2N},\omega^*_{2N})\right)^T.$$

Assume that $\sum_{j=1}^N \ell_j L_j = 0$ with $l_j \in \mathbb{R}$. Then, for all $i=1,\dots,N$, we have

$$\sum_{j=1}^N \ell_j f^X_{I_j}(\omega^1_{1i},\dots,\omega^{m_1}_{1i},\omega^1_{2i},\dots,\omega^{m_2-1}_{2i},\omega^*_{2i}) = 0. \tag{3.93}$$

By Lemma 3.25, we have $l_j = 0$ for $j=1,\dots,N$. Then, the family $(L_j)_{j=1,\dots,N}$ is linearly independent. We conclude that det A is not equal to zero. This ends the proof of Corollary 3.26.

□

We still need another technical lemma which guarantees that there exist integer frequencies such that Eq. (3.51) is satisfied and the matrix A in Eq. (3.62) is invertible.

Lemma 3.27. *There exists integer frequencies such that* (3.51) *is satisfied and det A is not equal to zero.*

Proof of Lemma 3.27. For $j=1,\dots,N$, we set

$$f_j(\omega_1,\dots,\omega_{m-1}) = f^X_{I_j}(\omega_1,\dots,\omega_{m-1},\sum_{i=1}^{m-1}\omega_i), \tag{3.94}$$

then, we have

$$\begin{aligned} \det A &= \begin{vmatrix} f_1(\omega_{11}^1,\dots,\omega_{21}^{m_2-1}), & \dots & ,f_1(\omega_{1N}^1,\dots,\omega_{2N}^{m_2-1}) \\ \vdots & \ddots & \vdots \\ f_N(\omega_{11}^1,\dots,\omega_{21}^{m_2-1}), & \dots & ,f_N(\omega_{1N}^1,\dots,\omega_{2N}^{m_2-1}) \end{vmatrix} \\ &= \frac{P(\omega_{11}^1,\dots,\omega_{2N}^{m_2-1})}{Q(\omega_{11}^1,\dots,\omega_{2N}^{m_2-1})}, \end{aligned} \tag{3.95}$$

where P and Q are two polynomials of $(m-1)N$ variables.

We first note that Q never vanishes over integer frequencies. Assume, by contradiction, that P is always equal to zero for integer frequencies verifying Eq. (3.51). Consider P as a polynomial in one variable $\omega_{2N}^{m_2-1}$, i.e.,

$$P(\omega_{11}^1,\dots,\omega_{2N}^{m_2-1}) = \sum_{j=0}^{M} P_j(\omega_{11}^1,\dots,\omega_{2N}^{m_2-2})(\omega_{2N}^{m_2-1})^j. \tag{3.96}$$

Given integer frequencies $(\omega_{11}^1,\dots,\omega_{2N}^{m_2-2})$, if Eq. (3.96) is not identically equal to zero, then this polynomial in the variable $\omega_{2N}^{m_2-1}$ most has a finite number of roots. However, for a given choice of $(m-1)N-1$ first frequencies, there exist an infinite number of $\omega_{2N}^{m_2-1}$ verifying (3.51). Then, $P_j = 0$ over all integer frequencies, and P_M is not identically equal to zero. We note that all P_j are polynomials of $(m-1)N-1$ variables. Proceeding by induction on the number of variables, it is easy to see that, at the end, we obtain a polynomial in the variable ω_{11}^1 which is equal to zero over all integer ω_{11}^1, and which is not identically equal to zero according to Corollary 3.26. That contradiction ends the proof of Lemma 3.27.

□

3.5.2.3 General case : $m > 2$

Notice that the proof of Theorem 3.21 does not really depend on the number of vector fields involved in the control system (4.5). Indeed, for $m > 2$, if the control functions are linear combination of sinusoids with integer frequencies, then the state variables in the canonical form are also linear combinations of sinusoids so that the frequencies are $\mathbb{Z}-$linear combinations of the frequencies occurring in the control functions. The proof is the same as that of Lemma 3.22, up to extra notation. Since Lemma 11 depends only on the length of the Lie brackets, but not on the number of vector fields, the proof of Lemma 3.25 does not depend on m, either. In order to prove a similar result for $m > 2$, we just need to re-project Eqs. (3.83) and (3.84) to m vector fields instead of 2.

3.5.3 Exact and sub-optimal steering law

In this section, we explain how we can devise, from Proposition 3.21, an *exact* and *sub-optimal* steering law (cf. Definition 3.17) $\text{Exact}_{m,r}$ for the approximate system, which is already in the canonical form and how $\text{Exact}_{m,r}$ can be incorporated into the *global approximate steering algorithm* (cf. Section 3.4.2). Note that $\text{Exact}_{m,r}$ only depends on the number of controlled vector fields m and on the maximum degree of nonholonomy r.

Recall that the components of $x \in \mathbb{R}^{\widetilde{n}^r}$ are partitioned into equivalence classes $\{\mathcal{E}_x^1, \mathcal{E}_x^2, \dots, \mathcal{E}_x^{\widetilde{N}}\}$ according to Definition 3.29 in such a way that $\mathcal{E}_x^i \prec \mathcal{E}_x^j$ for $(i,j) \in \{1,\dots,\widetilde{N}\}^2$ and $i < j$. For every equivalence class $\mathcal{E}_x^j$, Proposition 3.21 and Subsection 3.5.2.3 guarantee that we can choose frequencies such that the corresponding control matrix A_j (cf. Definition 3.32)

is invertible and the corresponding control function u^j obtained by Eq. (3.52) steers all the elements of $\mathcal{E}_x^j$ from an arbitrary initial value to the origin 0 (see Remark 3.22) without modifying any elements belonging to smaller classes.

Let $x^{\text{initial}} \in \mathbb{R}^{\widetilde{n}^r}$. Let $B_j := A_j^{-1}$ and $N_j := \text{Card}(\mathcal{E}_x^j)$, $j = 1, \ldots, \widetilde{N}$. For $x \in \mathbb{R}^{\widetilde{n}^r}$, we will use $[x]_{i,\ldots,k}$ with $1 \leq i < k \leq \widetilde{n}_r$ to denote the vector $(x_i, \ldots, x_k)$, and $\|x\|$ to denote the *pseudo-norm* of x defined by the *free* weights (cf. Definition 3.13 and Definition 3.26). We also define an intermediate function Position(u) as follows : if System (3.40) starts from $x = 0$ and evolves under the action of u, then Position(u) returns its position at $t = 2\pi$.

Algorithm 8 Exact Steering Law : $\text{Exact}_{m,r}(x^{\text{initial}})$

Require: $B_1, \ldots, B_{\widetilde{N}}$, and $N_1, \ldots, N_{\widetilde{N}}$;

1: $\lambda := \|x^{\text{initial}}\|_0$;
2: $x^{\text{new}} := \delta_{0,\frac{1}{\lambda}}(x^{\text{initial}})$;
3: $\hat{u}_{\text{norm}} := 0$;
4: $i := 0$;
5: **for** $j = 1, \ldots, \widetilde{N}$ **do**
6: $\quad x := [x^{\text{new}}]_{i+1,\ldots,i+N_j}$;
7: $\quad a^j := B_j\ x$;
8: $\quad$ construct u^j from a^j by Eq. (3.52) ;
9: $\quad x^{\text{new}} := x^{\text{new}} + \text{Position}(u^j)$;
10: $\quad \hat{u}_{\text{norm}} := \hat{u}_{\text{norm}} * u^j$ (cf. Definition 3.31) ;
11: $\quad i = i + N_j$;
12: **return** $\hat{u} := \lambda \hat{u}_{\text{norm}}$.

Proposition 3.28. *For every $x^{\text{initial}} \in \mathbb{R}^{\widetilde{n}^r}$, the input given by* $\text{Exact}_{m,r}(x^{\text{initial}})$ *steers System (3.40) from x^{initial} to 0 exactly. Moreover, there exists a constant $C > 0$ such that*

$$\ell(\text{Exact}_{m,r}(x^{\text{initial}})) \leq C d(x^{\text{initial}}, 0), \qquad \forall\ x^{\text{initial}} \in \mathbb{R}^{\widetilde{n}^r}, \tag{3.97}$$

where we use d to denote the sub-Riemannian distance defined by the family X.

Proof of Proposition 3.28. The fact that the procedure described by the Lines $5 - 12$ in Algorithm 8 produces an input $\hat{u}_{\text{norm}}$ steering System (3.40) from $\delta_{0,\frac{1}{\lambda}}(x^{\text{initial}})$ to 0 is a consequence of Proposition 3.21 and Subsection 3.5.2.3. We also note that, due to the homogeneity of System (3.40), if an input u steers (3.40) from x to 0, then, for every $\lambda > 0$, the input λu steers (3.40) from $\delta_{0,\lambda}(x)$ to 0. Therefore, the input computed by $\text{Exact}_{m,r}(x^{\text{initial}})$ steers System (3.40) from x^{initial} to 0. Let us now show (3.97). In the sequel, the application $\text{Exact}_{m,r} : x \to \text{Exact}_{m,r}(x)$ will be simply denoted by $\hat{u} : x \to \hat{u}(x)$. Let $S(0,1) := \{y, \|y\|_0 = 1\}$ and $x \in \mathbb{R}^{\widetilde{n}^r}$. Then, there exists $x_{\text{norm}} \in S(0,1)$ such that $x = \delta_{0,\lambda}(x_{\text{norm}})$ with $\lambda := \|x\|_0$. We have :

$$\ell(\hat{u}(x)) = \ell(\lambda \hat{u}(x_{\text{norm}})) = \lambda \ell(\hat{u}(x_{\text{norm}})) \leq \lambda \sup_{y \in S(0,1)} \ell(\hat{u}(y)).$$

We also know that, since the sub-Riemannian distance $d(0,\cdot)$ from 0 and the pseudo-norm $\|\cdot\|_0$ at 0 are both homogeneous of degree 1 with respect to the dilation $\delta_{0,t}(\cdot)$, there exists a constant $\widetilde{C} > 0$ such that $\widetilde{C}\lambda \leq d(0,x)$. Since the application $y \to \hat{u}(y)$ is continuous from $S(0,1)$ to $\mathbb{R}^m$ and $S(0,1)$ is compact, then, $\sup_{y\in S(0,1)} \ell(\hat{u}(y))$ is bounded, thus the inequality (3.97) holds true.

□

The following theorem is a consequence of Proposition 3.28 and Remark 3.21.

Theorem 3.29. *The function* $\text{Exact}_{m,r}(\cdot)$ *constructed by* Algorithm 8 *provides the approximate system* $\mathcal{A}^X$ *defined in Section 3.4.1 with a sub-optimal steering law.*

Proof of Theorem 3.29. It suffices to note that, for every $a \in \Omega$, $\mathcal{A}^X(a)$ has the same form (cf. Remark 3.21), thus defines the same sub-Riemannian distance d. Therefore, the inequality (3.97) holds uniformly with respect to the approximate point a, and this terminates the proof of Theorem 3.29.

□

Remark 3.27. Frequencies choices and the construction of the corresponding control matrix A_j, as well as its inverse B_j, translate to off-line computations. We note that Proposition 3.21 only gives sufficient conditions to prevent resonance (by choosing widely spaced frequencies, cf. Eq. (3.51)) and guarantee the invertibility of the corresponding matrix (by using a sufficiently large number of independent frequencies). These conditions tend to produce high frequencies while it is desirable to find smaller ones for practical use. We can prove that two independent frequencies suffice to steer one component (cf. Section 3.5.2.1), and we *conjecture* that $2N$ independent frequencies suffice to control one equivalence class of cardinal N by producing an invertible matrix. One can implement a searching algorithm for finding the optimal frequencies for each equivalence class such that they prevent all resonances in smaller classes and produce an invertible matrix for the class under consideration. Proposition 3.21 guarantees the finiteness of such an algorithm. Moreover, one can construct, once for all, a table containing the choice of frequencies and the corresponding matrices for each equivalence class in the free canonical system.

Remark 3.28. Recall that the key point in our control strategy consists in choosing suitable frequencies such that, during each $2\pi-$period, the corresponding input function displaces components of one equivalence class to their preassigned positions while all the components of smaller classes (according to the ordering in Definition 3.30) return at the end of this control period to the values taken at the beginning of the period. In order to achieve the previous task, special resonance conditions must be verified by the appropriate components, and these conditions must not hold for all the other smaller components (according to the ordering in Definition 3.30). Note that two categories of frequencies have been picked in Proposition 3.21 : the basic frequencies $\{\omega_{ij}^k\}$, and the resonance frequencies $\{\omega_{ij}^*\}$. Since frequencies occurring in the dynamics of the state variables are just $\mathbb{Z}-$linear combinations of $\{\omega_{ij}^k\} \cup \{\omega_{ij}^*\}$, and the resonance frequencies $\{\omega_{ij}^*\}$ are chosen to be special $\mathbb{Z}-$linear combinations of basic frequencies (resonance condition), the frequencies in the dynamics of the state variables are special $\mathbb{Z}-$linear combinations of $\{\omega_{ij}^k\}$.

Remark 3.29. Once the frequencies and matrices are obtained, the on-line computation $\text{Exact}_{m,r}$ is only a series of matrix multiplications which can be performed on-line without any numerical difficulty.

Remark 3.30. The Desingularization Algorithm presented in Section 3.3.3 (see also Remarks 3.14 and 3.24) together with Algorithm 8 provides general nilpotent control systems with an exact steering method, which is also sub-optimal.

Remark 3.31. We note that the inputs constructed in this section are piecewise C^∞ during each time interval $[2i\pi, 2(i+1)\pi]$, for $i = 1, \ldots, \widetilde{N} - 1$, but they are not globally continuous during the entire control period $[0,\ 2\widetilde{N}\pi]$, due to discontinuity at $t = 2\pi, 4\pi, \ldots, 2(\widetilde{N} - 1)\pi$. However, it is not difficult to devise (globally) continuous inputs using interpolation techniques. We illustrate the idea with a simple example. Assume that we use $u^i = (u_1^i, u_2^i)$ and $u^j = (u_1^j, u_2^j)$ defined by

$$\begin{aligned} u_1^i(t) &= \cos\omega_{1i}t, \\ u_2^i(t) &= \cos\omega_{2i}t + a^i\cos(\omega_{2I}^*t + \varepsilon^i\frac{\pi}{2}), \qquad t \in [2(i-1)\pi, 2i\pi], \\ u_1^j(t) &= \cos\omega_{1j}t, \\ u_2^j(t) &= \cos\omega_{2j}t + a^j\cos(\omega_{2j}^*t + \varepsilon^j\frac{\pi}{2}), \qquad t \in [2(j-1)\pi, 2j\pi], \end{aligned}$$

to steer two consecutive classes $\mathcal{E}_x^i$ and $\mathcal{E}_x^j$ (i.e. $j = i+1$) which are both of cardinal equal to 1.

If we require their concatenation $u^i * u^j$ to be continuous, i.e.

$$u_1^i(2\pi) = u_1^j(2\pi), \tag{3.98}$$
$$u_2^i(2\pi) = u_2^j(2\pi), \tag{3.99}$$

we can proceed as follows.

For Eq. (3.98), it suffices to modify slightly u_1^j. We take

$$\widetilde{u}_1^j(t) = u_1^i(2\pi)\cos\omega_{1J}t. \tag{3.100}$$

For Eq. (3.99), we distinguish two cases :
– if $\varepsilon^j = 1$, we can take

$$\begin{aligned} \widetilde{u}_2^j(t) &= u_2^i(2\pi)\cos\omega_{2j}t + a^j\cos(\omega_{2j}^*t - \frac{\pi}{2}) \\ &= u_2^i(2\pi)\cos\omega_{2j}t + a^j\sin\omega_{2j}^*t; \end{aligned} \tag{3.101}$$

– if $\varepsilon^j = 0$, we add a frequency ω_c to u_2^j, large enough to avoid any additional resonances,

$$\widetilde{u}_2^j(t) = \cos\omega_{2j}t + a^j\cos\omega_{2j}^*t + (u_2^i(2\pi) - a^j - 1)\cos\omega_c t. \tag{3.102}$$

Let $\widetilde{u}^j := (\widetilde{u}_1^j,\ \widetilde{u}_2^j)$. Then, by construction, the new input $u^i * \widetilde{u}^j$ is continuous over the time interval $[2i\pi, 2j\pi]$.

It is clear that this idea of interpolation by adding suitable frequencies can be used to construct continuous inputs over the entire control period $[0, 2\widetilde{N}\pi]$. In fact, by using more refined interpolations, one can get inputs of class C^k for arbitrary finite integer k.

Remark 3.32. Using the idea presented in Remark 3.31 together with Remark 3.30, it is easy to conclude that, for general nilpotent systems, the resulting trajectories are globally C^1 curves and the regularity does not depend on the time-parameterization of the trajectories.

3.6 Appendix

For the sake of completeness, we give in this short appendix the proof of Theorem 3.1 together with some comments on Algorithm 5.

3.6.1 Proof of Theorem 3.1

We first note that Steps 1 through 5 in Algorithm 5 are straightforward.

Theorem 3.11 guarantees that the Desingularization Algorithm (Section 3.3.3) provides us with a new family of vectors fields $\xi = \{\xi_1, \ldots, \xi_m\}$, which is regular and free up to step r with r denoting the maximum value of the degree of nonholonomy of the original system $X = \{X_1, \ldots, X_m\}$, on the corresponding compact set $\mathcal{V}^c_{\mathcal{J}_i}$. Then, we construct the approximate system $\mathcal{A}^\xi$ using the procedure presented in Section 3.4.1 and provide it with the sub-optimal steering law $\text{Exact}_{m,r}$ defined in Algorithm 8. The sub-optimality of $\text{Exact}_{m,r}$ is guaranteed by Theorem 3.29. Therefore, by Theorem 3.15, the LAS method AppSteer associated with $\mathcal{A}^\xi$ and its steering law is uniformly contractive on the compact set $\mathcal{V}^c_{\mathcal{J}_i} \times \bar{B}_R(0)$. Then, by Theorem 3.16,
GlobalFree $(\widetilde{x}_0, \widetilde{x}_1, e, \mathcal{V}^c, \text{AppSteer})$ provided by Algorithm 6 terminates in a finite number of steps and stops at a point $\widetilde{x}$ such that $d(\widetilde{x}, \widetilde{x}_1) < e$. Since there is a finite number of compacts to be explored, we conclude that Algorithm 5 terminates in a finite number of steps and steers the system (Σ) from x^{initial} to some point x such that $d(x, x^{\text{final}}) < e$. This ends the proof of Theorem 3.1.

3.6.2 About the control set

Let $U \subset \mathbb{R}^m$ be any neighborhood of the origin. Then every trajectory of (4.5) corresponding to the inputs produced by Algorithm 5 can be time-reparameterized so that the resulting trajectory of (4.5) is associated with an input taking values in U.

3.6.3 Getting trajectories of class C^1 for the original control system

We can slightly modify Algorithm 5 to get trajectories of class C^1 for the original control system (Σ). This is equivalent to ask for continuous inputs produced by the algorithm. According to Remark 3.31, inputs can be made continuous within each iteration step in Algorithm 6 since they are computed based the nilpotent approximate system. By using the same interpolation technique as presented in Remark 3.31, we can still produce inputs which remain continuous from one step to another in Algorithm 6. Therefore, trajectories of class C^1 for the control system (Σ) are obtained.

Chapitre 4

A motion planning algorithm for the rolling-body problem

Le contenu de ce chapitre fait l'objet d'un article en collaboration avec F. Alouges et Y. Chitour, à paraître dans les *IEEE Transactions on Robotics*, volume 26 (2010), numéro 5 (Octobre).

Sommaire

4.1 Introduction

In recent years, nonholonomic systems have attracted much attention due to the theoretical questions raised for their motion planning and to their importance in numerous applications (cf. [69, 75] and references therein). In particular, the planning of robotic manipulators for achieving high operational capability with low constructive complexity is a major issue for the control community in the last decade. Nonholonomy is exploited for the design of such manipulators but ensuring both hardware reduction and controllability performances yields serious difficulties, requiring more elaborate analysis and efficient algorithm. The rolling-body problem illustrates well all the aforementioned aspects.

We recall that the rolling-body problem (without slipping or spinning) is a control system Σ modeling the rolling of a connected surface S_1 on another one S_2 of the Euclidean space $\mathbb{R}^3$ so that the relative speed of the contact point is zero (no slipping) and the relative angular velocity has zero component along the common normal direction at the contact point (no spinning). It is intuitively clear that five parameters are needed to describe the state of Σ : two for parameterizing the contact point as element of S_1, two others for the contact point as

element of S_2, and finally one more parameter for the relative orientation of S_1 with respect to S_2. Therefore, the state space $\mathcal{Q}(S_1, S_2)$ of Σ is a $5-$dimensional manifold and it can be shown that $\mathcal{Q}(S_1, S_2)$ is a circle bundle over $S_1 \times S_2$. Because of the rolling constraints (no slipping, no spinning), one easily shows that, once an absolutely continuous (a.c. for short) curve c_1 on S_1 is prescribed, there exists a unique a.c. curve Γ in $\mathcal{Q}$ describing the rolling without slipping or spinning of the surface S_2 onto the surface S_1 along the curve c_1. Thus, the admissible inputs of the control system Σ exactly correspond to the a.c. curves c_1 of S_1 by their velocities $\dot{c}_1$. As a consequence, Σ can be written (in local coordinates) as a driftless control system of the type $\dot{x} = u_1 F_1(x) + u_2 F_2(x)$, where $(u_1, u_2) \in \mathbb{R}^2$ is the control and F_1, F_2 are vector fields defined in the domain of the chart (cf. [75, 69] and references therein). As regards controllability issues, there exist several works (cf. [69] and references therein) addressing these questions. Agrachev and Sachkov (cf. [2]) proved that Σ is completely controllable if and only if S_1 and S_2 are not isometric. Marigo and Bicchi (cf. [69]) provided geometric descriptions for the possible reachable sets. One of the main conclusions of these works will be instrumental for us and goes as follows : the control system Σ is locally controllable at a point $q \in \mathcal{Q}$ if $K_{S_1}(\mathrm{pr}_1(q)) - K_{S_2}(\mathrm{pr}_2(q)) \neq 0$, where $K_{S_1}(\cdot)$ and $K_{S_2}(\cdot)$ respectively denote the Gaussian curvature of S_1 and S_2, and $\mathrm{pr}_i : \mathcal{Q} \to S_i$, $i = 1, 2$, are the canonical projections. In particular, if S_1 is a strictly convex surface (i.e. $K_{S_1}(q_1) > 0$ for all $q_1 \in S_1$) and $S_2 = \mathbb{R}^2$, then the control system (Σ) is not only completely controllable, but also locally controllable at every point $q \in \mathcal{Q}$. On the opposite direction, it is worth mentioning the following result : Σ is not completely controllable if and only if S_1 and S_2 are isometric, with an isometry of $\mathbb{R}^3$.

Regarding the motion planning problem (MPP for short) associated to the rolling-body problem, most of the attention focused on the rolling of a convex surface S on a flat one, due to the fact that the latter models dexterous robotic manipulation of a convex object by means of a robotic hand with as few as three motors and flat finger, see [69, 75] and references therein. Moreover, in [69], several prototype dexterous grippers are exhibited. Recall that the MPP is the problem of finding a procedure that, for every pair (p, q) of the state space of a control system Σ, *effectively* produces a control $u_{p,q}$ giving rise to an admissible trajectory steering p to q. Note that in the category of rolling-body problem, even the simplest model, the so-called plate-ball system (a sphere rolling on the plane), does not allow any chained-form transformation and is not a flat system. We can hierarchize this category of problems as follows according to increasing level of difficulty :

L1. S_1 rolling on the plane :
- L1-1. the plane is free of prohibited regions ;
- L1-2. there are prohibited regions (obstacles) on the plane ;
- L1-3. there are prohibited regions on S_1 ;

L2. S_1 rolling on the top of S_2 with S_2 non flat, S_1 and S_2 non isometric :
- L2-1. there are prohibited regions neither on S_1 nor on S_2 ;
- L2-2. there are prohibited regions on S_1 or (and) S_2.

For **L1.**, there exists essentially one family of methods commonly called geometric phase methods based on the Gauss-Bonnet Theorem in differential geometry and initiated by Li and Canny. In [66], Li and Canny proposed a first general framework for solving **L1-1**. They devised an ingenious algorithm solving efficiently the MPP of plate-ball problem. However, their method cannot be directly applied to more general convex surfaces S_1 since explicit computation of the integral of the Gaussian curvature over a bounded region on S is in general not available. In the spirit of [66], Bicchi and Marigo proposed in [15] an *approximate* motion planning algorithm solving **L1-1** and **L1-2** for *general* convex body S_1. By using a

lattice structure on the state space, they translated Li-Canny's global and exact computation into a series of local and approximate ones (basic actions), easier to treat in practice. They also showed that this approximate method has good topological properties so that it can be incorporated into a more general motion planning algorithm dealing with obstacles in the plane. However, since a fine-grid lattice is needed in order to improve the precision, a large number of *periodical* maneuvers is necessary for achieving the preassigned change of orientation, producing thus highly oscillating-type motions, which may not be desirable in practice.

In [23], two other approaches to solve **L1-1** were proposed. The first one is based on the Liouvillian character of Σ. One can show that, if S_1 admits a symmetry of revolution, the MPP can be reduced to a purely inverse algebraic problem. However, such an approach presents a serious numerical drawback : the resulting inverse problem requires that implicit functions must be determined through transcendental equations involving local charts for S_1. The second approach proposed in [23] is based on the well-known continuation method (also called homotopy method or continuous Newton's algorithm [4]) which dates back to Poincaré. The MPP is therefore addressed as a pure inverse problem. Let us briefly recall how the continuation method (CM for short) works. It is used for solving nonlinear equations of the form $F(x) = y$, where x is the unknown and $F : X \to Y$ is surjective. Consider $x_0 \in X$ and $y_0 = F(x_0)$. Pick a differentiable path $\pi : [0,1] \to Y$ joining y_0 to the given y. Then, the CM is an iterative procedure which lifts π to a path $\Pi : [0,1] \to X$ so that $F \circ \Pi = \pi$. The word "iterative" refers to the fact that the path Π is obtained by the flow of a differential equation defined on X. Indeed, one starts by differentiating $F(\Pi(s)) = \pi(s)$ to get $DF(\Pi(s))\dot{\Pi}(s) = \dot{\pi}(s)$. The latter is satisfied by setting $\dot{\Pi}(s) := P(\Pi(s))\dot{\pi}(s)$, where $P(x)$ is a right inverse of $DF(x)$. Therefore, solving $F(x) = y$ amounts to first show that $P(\Pi(s))$ exists (for instance if $DF(\Pi(s))$ is surjective) and second to prove that the ODE in X, $\dot{\Pi}(s) = P(\Pi(s))\dot{\pi}(s)$, which is a "highly" non-linear equation (also called the Path Lifting Equation or Wazewski Equation [108]), admits a global solution. In the context of the MPP, the CM was introduced in [35] and [96, 97], and further developed in [25, 24, 29, 102, 103]. The map F is now an end-point map from the space of admissible inputs to the state space. Its singularities are exactly the abnormal extremals of the sub-Riemannian metric induced by the dynamics of the system, which are usually a major obstacle for the CM to apply efficiently to the MPP. In the case of Σ, non trivial abnormal extremals and their trajectories were determined in [23] and they exactly correspond to the horizontal geodesics of Σ. Despite that obstacle, assuming that the surface S is strictly convex and possesses a stable periodic geodesic, it was shown in [23] that the CM provides complete answers to the MPP. More precisely, it was shown that there exist enough paths π in the state space of Σ that can be lifted to paths Π in the control space by showing global existence of solutions to the Wazewski equation.

In this paper, we provide full details for the numerical implementation of the continuation method presented above in order to solve efficiently **L1-1**. The paper is organized as follows : in Section 4.2, we present the kinematic equations of motion of a convex body S_1 *rolling without slipping or spinning* on top of another one S_2. We describe in Section 4.3 how the continuation method can be applied to the motion planning problem. Sufficient conditions guaranteeing the existence of $P(\Pi)$ and the existence of a global solution of the Path Lifting Equation in the case of the rolling-body problem are also reported. Section 4.4 serves to detail some key points for numerical resolution of Path Lifting Equation. In Section 4.5, several numerical simulations are presented. Some detailed comments and possible generalizations will be presented at the end of this paper in Section 4.6.

4.2 Description of the rolling-body problem

In this section, we briefly recall how to derive the equations of motion for the rolling-body problem with no slipping or spinning of a connected surface S_1 of the Euclidean space $\mathbb{R}^3$ on top of another one S_2. This section does not bring new results but we provide it for sake of completeness and also to exhibit the numerical challenges raised by trying to implement ordinary differential equations on a manifold. These results were already obtained in [2, 69, 75].

We start by the intrinsic formulation of the problem, i.e., we first assume that S_1 and S_2 are two-dimensional, connected, oriented, smooth, complete Riemannian manifolds.

4.2.1 Differential geometric notions and definitions

If P is a matrix, we use P^T and $tr(P)$ to denote respectively the transpose of P, and its the trace.

Let $(S, \langle\cdot,\cdot\rangle)$ be a two-dimensional, connected, oriented smooth complete Riemannian manifold for the Riemannian metric $\langle\cdot,\cdot\rangle$. We use TS to denote the tangent bundle over S and US the unit tangent bundle, i.e. the subset of TS of points (x, v) such that $x \in S$ and $v \in T_xS$, $\langle v, v\rangle = 1$.

Let $\{U_\alpha, \alpha\}_{\alpha\in\mathcal{A}}$ be an atlas on S. For $\alpha, \beta \in \mathcal{A}$ such that $U_\alpha \cap U_\beta$ is not empty, we denote by $J_{\beta\alpha}$ the jacobian matrix of $\varphi^\beta \circ (\varphi^\alpha)^{-1}$ the coordinate transformation on $\varphi^\alpha(U_\alpha \cap U_\beta)$. For $\alpha \in \mathcal{A}$, the Riemannian metric is represented by the symmetric definite positive matrix $\mathcal{I}^\alpha$ and set $M^\alpha := \sqrt{\mathcal{I}^\alpha}$.

For $x \in S$, a frame f at x is an ordered basis for T_xS and, for $\alpha, \beta \in \mathcal{A}$, we have $f^\beta = J_{\beta\alpha} f^\alpha$. The frame f is orthonormal if, in addition $M^\alpha f^\alpha$ is an orthogonal matrix. An Orthonormal Moving Frame (briefly OMF) defined on an open subset U of S is a smooth map assigning to each $x \in U$ a positively oriented orthonormal frame $f(x)$ of T_xS.

Let ∇ be the Riemannain connection on S (cf. [91]). For a given OMF f defined on $U \subset S$, the Christoffel symbols associated to $f = (f_1, f_2)$ are defined by

$$\nabla_{f_i} f_j = \sum_k \Gamma_{ij}^k f_k,$$

where $1 \le i, j, k \le 2$. The connection form ω is the mapping defined on U such that, for every $x \in U$, ω_x is the linear application from T_xS to the set of 2×2 skew-symmetric matrices given as follows. For $i, j, k = 1, 2$, the $(i, j)-$th coefficient of $\omega_x(f_k)$ is equal to Γ_{ij}^k.

Let $c : J \to S$ be an absolutely continuous curve in S with J compact interval of $\mathbb{R}$. Set $X(t) := \dot{c}(t)$ in J which defines a vector field along c. Let $Y : J \to TS$ be an absolutely continuous assignment such that, for every $t \in J$, $Y(t) \in T_{c(t)}S$. We say that Y is parallel along c if $\nabla_X Y = 0$ for almost all $t \in J$. In the domain of an OMF f, that equation can be written as follows

$$\dot{Y}^k = -\sum_{1\le i,j\le 2} \Gamma_{ij}^k X^i Y^j,$$

or equivalently,

$$\dot{Y} = -\omega(X)Y.$$

Recall that a curve c is a geodesic if the velocity $\dot{c}(t)$ is parallel along c, that is

$$\nabla_{\dot{c}}\dot{c} = 0. \tag{4.1}$$

4.2.2 Rolling body problem

4.2.2.1 Definition of the state space

Consider now the rolling-body problem with no slipping or spinning of S_1 on top of S_2. We adopt here the viewpoint presented in [2].

At the contact points of the bodies $x_1 \in S_1$ and $x_2 \in S_2$, their tangent spaces are identified by an orientation-preserving isometry

$$q : T_{x_1}S_1 \longrightarrow T_{x_2}S_2,$$

Such an isometry q is a state of the system, and the state space is

$$\begin{aligned} & \mathcal{Q}(S_1, S_2) \\ = \; & \{q : T_{x_1}S_1 \to T_{x_2}S_2 \mid x_1 \in S_1, x_2 \in S_2, q \text{ isometry}\}. \end{aligned}$$

As the set of all orientation-preserving isometries in $\mathbb{R}^2$ is $SO(2)$, which can be identified with the unit circle S^1 in $\mathbb{R}^2$, $\mathcal{Q}(S_1, S_2)$ is a 5-dimensional connected manifold. A point $q \in \mathcal{Q}(S_1, S_2)$ is locally parametrized by $(x_1,\ x_2,\ R)$ with $x_1 \in S_1$, $x_2 \in S_2$ and $R \in SO(2)$.

4.2.2.2 Rolling dynamics

We next describe the motion of one body rolling on top of another one so that the contact point of the first follows a prescribed absolutely continuous (a.c. for short) curve on the second body.

Let f_1 and f_2 be two OMFs defined on the chart domains of α_1, α_2. For $i = 1, 2$, consider a curve $c_i^{\alpha_i}$ defined inside the chart domain α_i on the body S_i. Let $b_i(t) = f_i(c_i(t))R_i(t)$ parallel along $c_i^{\alpha_i}$, $i = 1, 2$, and $R := R_2(t)R_1(t)^{-1} \in SO(2)$ which, by definition, measures the relative position of f_2 with respect to f_1 along $(c_1^{\alpha_1}, c_2^{\alpha_2})$. The variation of R_i along $c_i^{\alpha_i}$, for $i = 1, 2$, is given by $\dot{R}_i = -\omega_i(\dot{c}_i^{\alpha_i})R_i$.

Given an a.c. curve $c_1 : [0, T] \to S_1$, the rolling of S_2 on S_1 without slipping or spinning along c_1 is characterized by a curve $\Gamma = (c_1, c_2, R) : [0, T] \to \mathcal{Q}(S_1, S_2)$ defined the two following conditions.

Up to initial conditions, the no slipping condition amounts to

$$M^{\alpha_2}\dot{c}_2^{\alpha_2}(t) = RM^{\alpha_1}\dot{c}_1^{\alpha_1}(t), \tag{4.2}$$

and the no spinning one to

$$\dot{R}R^{-1} = R\omega_1(\dot{c}_1^{\alpha_1})R^{-1} - \omega_2(\dot{c}_2^{\alpha_2}). \tag{4.3}$$

Since $SO(2)$ is commutative, equation (4.3) reduces to

$$\dot{R}R^{-1} = \omega_1(\dot{c}_1^{\alpha_1}) - \omega_2(\dot{c}_2^{\alpha_2}). \tag{4.4}$$

If we fix a point $x = (x_1, x_2, R_0) \in \mathcal{Q}(S_1, S_2)$, a curve c_1 on S_1 starting at x_1 defines entirely the curve Γ by equations (4.2) and (4.4). Therefore, we can give the following definition :

Definition 4.1. The surface S_2 rolls on the surface S_1 without slipping or spinning if, for every $x = (x_1, x_2, R_0) \in \mathcal{Q}(S_1, S_2)$ and a.c. curve $c_1 : [0, T] \to S_1$ starting at x_1, there exists an a.c. curve $\Gamma : [0, T] \to \mathcal{Q}(S_1, S_2)$ with $\Gamma(t) = (c_1(t), c_2(t), R(t))$, $\Gamma(0) = x$ and for every $t \in [0, T]$, such that, on appropriate charts, equations (4.2) and (4.4) are satisfied. We call the curve $\Gamma(t)$ an admissible trajectory.

If we consider f_1 and f_2 two OMFs and if the state x is represented (in coordinates) by the triple $x = (c_1, c_2, R)$, then for almost all t such that $x(t)$ remains in the domain of an appropriate chart, there exists a measurable function $u(\cdot)$ (called control) with values in $\mathbb{R}^2$ such that

$$\begin{aligned}
\dot{c_1}(t) &= u_1(t) f_1^1(c_1(t)) + u_2(t) f_2^1(c_1(t)), \\
\dot{c_2}(t) &= u_1(t)(f^2(c_2(t))R(t))_1 \\
&\quad + u_2(t)(f^2(c_2(t))R(t))_2, \\
\dot{R}(t)R^{-1}(t) &= \sum_{i=1}^{2} u_i(t)[\omega_1(f_i^1(c_1(t))) \\
&\quad - \omega_2(f^2(c_2(t))R(t))_i].
\end{aligned}$$

Let us consider the vector fields F_1 and F_2 defined by

$$F_i = (f_i^1,\ (f^2 R)_i,\ [\omega_1(f_i^1) - \omega_2(f^2 R)_i])^T, \quad i = 1, 2.$$

Then, the previous system of equations have the following compact form in local coordinates,

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x). \tag{4.5}$$

We recognize the classical form of a driftless control-affine system.

Remark 4.1. In general, it is not possible to get a global basis for the distribution Δ and thus to define globally the dynamics of the control system using vector fields. One notable exception occurs when one of the manifolds is a plane, cf. [23]. Therefore, addressing the motion planning efficiently (i.e. as far as producing a numerical scheme) becomes a delicate issue since most of the standard techniques are based on global vector field expressions of the dynamics of a control system.

The following proposition describes a fundamental property of the rolling problem. For more detail, see [23] for instance.

Proposition 4.1. *Let $u \in H$ be an admissible control that gives rise to the admissible trajectory*

$$\Gamma = (c_1, c_2, R) :\ [0, 1] \to M.$$

Then the following statements are equivalent :

(a) *the curve $c_1 : [0, 1] \to S_1$ is a geodesic ;*
(b) *the curve $c_2 : [0, 1] \to S_2$ is a geodesic ;*
(c) *the curve $\Gamma : [0, 1] \to M$ is a horizontal geodesic.*

Remark 4.2. In the case where S_2 is a plane, if S_1 is rolling along a piecewise linear curve c_2 defined on S_2, then, Proposition 4.1 allows us to construct the locus of the contact point on S_1. Indeed, since S_2 is flat, c_2 is piecewise *geodesic*, it suffices then to integrate a geodesic equation on S_1 to get the locus of the contact point. See Subsection 4.4.3 for more details.

4.2.2.3 Rolling body problem in $\mathbb{R}^3$

From now on, we will assume that the manifolds S_1 and S_2 are oriented surfaces of $\mathbb{R}^3$ with metrics induced by the Euclidean metric of $\mathbb{R}^3$.

We first note that there are two possible ways to define the rolling problem, depending on the respective (global) choice of normal vectors for S_1 and S_2. Indeed, the orientation of the tangents planes of an oriented surface S is determined by the choice of a Gauss map i.e. a continuous normal vector $n : S \to S^2$, with S^2 denoting the sphere of radius 1 in $\mathbb{R}^3$. There are two such normal vectors, n and $-n$. If S is (strictly) convex, these two normal vectors are called inward and outward.

Recall that the rolling-body problem assumes that the tangent spaces at the contact points are identified. In $\mathbb{R}^3$, this is equivalent to identify the normal vectors. Let n_i be the normal vector of S_i, then at contact points, we can either assign n_1 to n_2 or $-n_2$, i.e. we have $n_1 = \varepsilon n_2$ with $\varepsilon = \pm 1$. The physical meaning of this parameter ε is the following : if $\varepsilon = 1$, the two surfaces roll so that one is "inside" the other one, in other words, they are on the same side of their common tangent space at the contact point ; if $\varepsilon = -1$, the two surfaces roll so that one is "outside" the other one, in other words, they are on opposite sides with respect to their common tangent space at the contact point. It is clear that the second situation is more physically feasible in general since, it holds true globally as soon as the two surfaces are convex. We will only deal with this second situation.

We note that Eq. (4.5) has simpler expression in geodesic coordinates. Recall that the geodesic coordinates on a Riemannian manifold S are charts (v, w) defined such that the matrix $\mathcal{I}^\alpha$ is diagonal and equal to $\mathrm{diag}(1, B^2(v, w))$. The function B is defined in an open neighborhood of $(0, 0)$ (the domain of the chart) and satisfies $B(0, w) = 1$, $B_v(0, w) = 0$ and $B_{vv} + K\ B = 0$, where K denotes the Gaussian curvature of S at (v, w) and B_v (B_{vv}, respectively) is the (double, respectively) partial derivative of B with respect to v.

Using the fact that $\mathcal{Q}(S_1, S_2)$ is a circle bundle when S_1 and S_2 are two-dimensional manifolds, and taking geodesic coordinates B_1, B_2 for S_1 and S_2 at contact points x_1 and x_2 respectively, consider coordinates $x = (v_1, w_1, v_2, w_2, \psi)$ in some neighborhood of $(0, \psi_0)$ in $\mathbb{R}^4 \times S^1$. Then, the control system (4.5) can be written locally as

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x), \tag{4.6}$$

with

$$F_1(x) = (1,\ 0,\ \cos\psi,\ -\frac{\sin\psi}{B_2},\ -\frac{B_{2v_2}}{B_2}\sin\psi)^T, \tag{4.7}$$

$$F_2(x) = (0,\ \frac{1}{B_1},\ -\sin\psi,\ -\frac{\cos\psi}{B_2},\ -\frac{B_{1v_1}}{B_1} - \frac{B_{2v_2}}{B_2}\cos\psi)^T, \tag{4.8}$$

see [23] for instance.

Remark 4.3. Since the functions B_1 and B_2 involving in the geodesic coordinates are obtained by solving partial differential equations, the rolling-dynamics given by Eqs. (4.7) and (4.8) is not completely explicit, thus it may not be suitable for numerical implementations. We will explain in Section 4.4 how to overcome this difficulty.

4.3 Continuation method

We start with a general description of the CM, see [25] for more details and complete justifications. The state space $\mathcal{Q}(S_1, S_2)$ is simply denoted by M. The admissible inputs u are elements of $H = L^2([0,1], \mathbb{R}^2)$. We use $\|u(t)\|$ and $\|u\|_H$ respectively to denote $(\sum_{i=1}^2 u_i^2(t))^{1/2}$ and $(\int_0^1 \|u(t)\|^2 dt)^{1/2}$. If $u, v \in H$, then $(u,v)_H = \int_0^1 u^T(t)v(t)dt$.

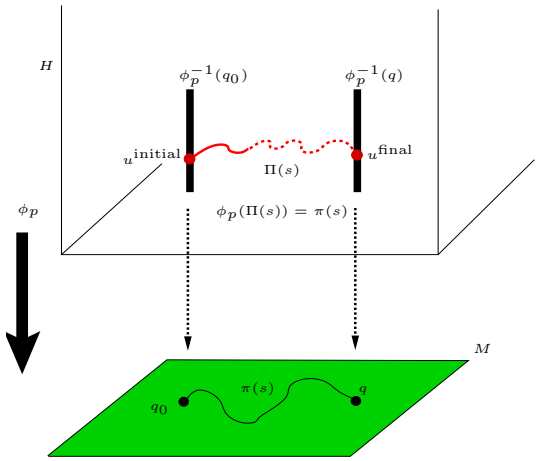
From the brief description of the continuation method given in the introduction, the map F is equal to the end-point $\phi_p : H \to M$ associated to some fixed $p \in M$. (For more details and complete justifications regarding the continuation method cf. [25].) For $u \in H$ and $p \in M$, let $\gamma_{p,u}$ be the trajectory of Σ starting at p for $t = 0$ and corresponding to u. Then, for $v \in H$, $\phi_p(v)$ is given by

$$\phi_p(v) := \gamma_{p,v}(1).$$

Recall that $\phi_p(v)$ is defined for every $v \in H$. The MPP can be reformulated as follows : for every $p, q \in M$, exhibit a control $u_{p,q} \in H$ such that

$$\phi_p(u_{p,q}) = q. \tag{4.9}$$

In other words, we want to inverse the end-point map ϕ_p, or more precisely, we are looking for a *right-inverse* of ϕ_p as this map is surjective (by the controllability assumption) but not injective ($u_{p,q}$ is not unique). This inversion is performed by using the continuation method summarized in the following scheme.



We start with an arbitrary control u^{initial}. Set $q_0 := \phi_p(u^{\text{initial}})$ and choose a path $\pi : [0,1] \to M$ such that $\pi(0) := q_0$ and $\pi(1) := q$. We now look for a path $\Pi : [0,1] \to H$ such that, for every $s \in [0,1]$,

$$\phi_p(\Pi(s)) = \pi(s). \tag{4.10}$$

Differentiating Eq. (4.10) yields to

$$D\phi_p(\Pi(s)) \cdot \frac{d\Pi}{ds}(s) = \frac{d\pi}{ds}(s). \tag{4.11}$$

If $D\phi_p(\Pi(s))$ has full rank, then Eq. (4.11) can be solved for $\Pi(s)$ by taking Π such that

$$\frac{d\Pi}{ds}(s) = P\big(\Pi(s)\big) \cdot \frac{d\pi}{ds}(s), \tag{4.12}$$

where $P(v)$ is a right inverse of $D\phi_p(v)$. For instance, we can choose $P(v)$ to be the Moore-Penrose pseudo-inverse of $D\phi_p(v)$.

We are then led to study the Wazewski equation (4.12) called the Path Lifting Equation (PLE) as an ODE in H. Recall that, by construction, the control defined by $u^{\text{final}} := \Pi(1)$ steers the system from p to q. In order to get the value of $\Pi(1)$, it suffices, at least formally, to solve the following initial value problem defined in the control space H :

$$\begin{cases} \dfrac{d\Pi}{ds}(s) &= P(\Pi(s)) \cdot \dfrac{d\pi}{ds}(s), \\ \Pi(0) &= u^{\text{initial}}. \end{cases} \tag{4.13}$$

Therefore, to successfully apply the CM to the MPP, we have to resolve two issues :

(a) non degeneracy : the path π has to be chosen so that, for every $s \in [0,1]$, $D\phi_p\big(\Pi(s)\big)$ has always full rank ;

(b) non explosion : to solve Eq. (4.9), the PLE defined in Eq. (4.12) must have a global solution on $[0,1]$.

Remark 4.4. Point (a) guarantees the existence of $P(\Pi(s))$ for every $s \in [0,1]$ so that Eq. (4.12) is always well defined. Point (b) is also important since we need to evaluate $\Pi(1)$ to get a control steering the system from p to q.

Remark 4.5. We note that local existence and uniqueness of the solution of the PLE hold as soon as ϕ_p is of class C^2.

It is reasonable to expect difficulties with the singular points of ϕ_p, *i.e.*, the controls $v \in H$ where rank $D\phi_p(v) < 5$ (cf. [18, 72, 27, 28] for general properties of singular points of the end-point map). Let S_p and $\phi_p(S_p)$ be the set of singular points of ϕ_p and the set of singular values respectively. The application of the CM to the MPP is thus decomposed in two steps. In the first one, we have to characterize (when possible) S_p and $\phi_p(S_p)$. The second step consists of lifting paths $\pi : [0,1] \to M$ avoiding $\phi_p(S_p)$ to paths $\Pi : [0,1] \to H$ globally defined on $[0,1]$ by Eq. (4.12).

A sufficient condition resolving (a) and (b) is given by

Condition 4.1. *We say that a closed subset $\mathcal{K}$ of M verifies Condition 4.1 if*

(i) $\mathcal{K}$ is disjoint from $\overline{\phi_p(S_p)}$, where $\overline{\phi_p(S_p)}$ is the closure of $\phi_p(S_p)$;

(ii) there exists $c_{\mathcal{K}} > 0$ such that, for every $u \in H$ with $\phi_p(u) \in \mathcal{K}$, we have

$$\|P(u)\| \leq c_{\mathcal{K}} \|u\|, \tag{4.14}$$

where

$$\|P(u)\| = \Big(\inf_{\|z\|=1} z^T D\phi_p(u) D\phi_p(u)^T z\Big)^{-1/2},$$

with $z \in T^*_{\phi_p(u)}M$.

Once the existence of a closed set $\mathcal{K}$ verifying the **Condition 4.1** is guaranteed, an application of Gronwall Lemma yields that, for every path $\pi : [0,1] \to \mathcal{K}$ of class C^1 and every control $\bar{u} \in H$ such that $\phi_p(\bar{u}) = \pi(0)$, the solution of the PLE defined in Eq. (4.12) with initial condition $\bar{u}$ exists globally on the interval $[0,1]$.

We now consider the MPP of a strictly convex surface S_1 rolling on the on a plane. It is shown in [23] that if S_1 verifies a simple geometric property (see **Condition 4.2** in Appendix), then there exists a compact subset $\mathcal{K}$ in M verifying the **Condition 4.1**, which is large enough to completely resolve the MPP. The reader can refer to Appendix for a

summary of the results regarding the set $\mathcal{K}$ for the rolling-body problem, and a complete development on this issue can be found in [23].

For numerical purposes, we recall here the structure of S_p and $\phi_p(S_p)$ for the rolling-body problem. A proof can be found in [23] for instance.

Proposition 4.2. *For $p \in M$, one has*

$$S_p = \{(v\cos\theta, v\sin\theta) | v \in L^2([0,1],\mathbb{R}), \theta \in [0,2\pi]\},$$

and $\phi_p(S_p)$ is equal to the union of the end-points of all horizontal geodesics starting at p, i.e. all trajectories starting at p and corresponding to one control $u \in S_p$.

In other words, Proposition 4.2 states that singular controls in the case of convex surfaces rolling on the plane are exactly straight lines on the plane.

4.4 Numerical implementation

In this section, we describe how the continuation method can be implemented in order to solve numerically the MPP for rolling-bodies in the case where S_1 is a strictly convex surface of $\mathbb{R}^3$ and S_2 is the Euclidean plane $\mathbb{R}^2$. In that case, the dynamics of the control system is given in geodesic coordinates by

$$\begin{aligned}
\dot{v}_2 &= u_1, \\
\dot{w}_2 &= u_2, \\
\dot{v}_1 &= \cos\psi u_1 - \sin\psi u_2, \\
\dot{w}_1 &= -\frac{1}{B}\sin(\psi)u_1 - \frac{1}{B}\cos(\psi)u_2, \\
\dot{\psi} &= -\frac{B_{v_1}}{B}\sin(\psi)u_1 - \frac{B_{v_1}}{B}\cos(\psi)u_2,
\end{aligned} \tag{4.15}$$

where we use B to denote the function occurring in the definition of geodesic coordinates on S_1. Note that Eq. (4.15) is deduced from Eq. (4.6) by assuming that S_2 is flat.

For the sake of simplicity, we make assumption that S_1 is defined as one bounded connected component of the zero-level set of a smooth real-valued function $f : \mathbb{R}^3 \to \mathbb{R}$. The normal vector field to S_1 is denoted by $n : S_1 \to S^2$ and is given in that case by

$$\frac{\nabla f}{\|\nabla f\|},$$

where $\nabla f = (f_x, f_y, f_z)$ denotes the gradient vector of f. The Gaussian curvature of S_1 is denoted by K and we assume that $K_{min} := \min_{S_1} K > 0$. In addition, set $K_{max} := \max_{S_1} K$.

In the sequel, we still use H and M to denote respectively the control space and the state space of the control system defined by Eq. (4.15). From Section 4.3, we deduce the following motion planning algorithm which, for any pair $(p, q) \in M \times M$, produces an input u^{final} steering the control system (4.15) from p to q.

Note that the only difficulty in **Algorithm 1** is step (iii), requiring to solve numerically an ordinary differential equation defined on the control space H which is in general an infinite dimensional vector space. In the following paragraphs, we detail some key points for solving Eq. (4.16).

Algorithm 9 Motion Planning Algorithm

(i) Choose an arbitrary non singular control $u^{\text{initial}} \in H$ such that $\text{pr}_1(q_0) = \text{pr}_1(q)$ where $q_0 := \phi_p(u^{\text{initial}})$.

(ii) Define a curve $\pi : [0,1] \to M$ such that $\pi(0) := q_0$ and $\pi(1) := q$.

(iii) Solve numerically the following initial value problem

$$\begin{cases} \dfrac{d\Pi}{ds}(s) &= P(\Pi(s)) \cdot \dfrac{d\pi}{ds}(s), \\ \Pi(0) &= u^{\text{initial}}. \end{cases} \tag{4.16}$$

(iv) Set $u^{\text{final}} := \Pi(1)$.

4.4.1 Discretizing the control space H

We start by approximating the control space H which is an infinite dimensional vector space. Recall that in our case controls are just plane curves $c_2 : [0,1] \to \mathbb{R}^2$ such that $\dot{c}_2 = (u_1, u_2)$ for almost all $t \in [0,1]$. We divide the interval $[0,1]$ into N parts, and we approximate the control space H by the $2N$-dimensional subspace $\hat{H}$ of piecewise linear functions. Then, c_2 can be approximated by $\hat{c}_2$, the linear interpolation of $(c_2^1, \ldots, c_2^N)$ where $c_2^i = c_2(\frac{i}{N-1}) = (x_i, y_i)^T$. On each segment $[t_i, t_{i+1}] = [\frac{i}{N-1}, \frac{i+1}{N-1}]$, the approximate control $(\hat{u}_1^i, \hat{u}_2^i)^T$ is proportional to the vector $(x_{i+1} - x_i, y_{i+1} - y_i)^T$.

Remark 4.6. We have chosen the space of piecewise linear functions as the approximate control space for two reasons : (i) piecewise linear curves are easy to be implemented on the plane ; (ii) the corresponding trajectories on S_1 are also easy to be obtained by integrating some geodesic equations by using Proposition 4.1 (see also Remark 4.2), instead of Eq. (4.15) where the function B defining the geodesic coordinates is not given explicitly. This second point plays a crucial role in improving the efficiency of our method. More details will be given in Subsection 4.4.3.

Remark 4.7. We note that elements in $\hat{H}$ are *piecewise* linear functions with more than one piece, then they are not singular inputs. See also Proposition 4.2.

The Path Lifting Equation (4.12) tells us how we have to modify this piecewise constant control $(\hat{u}_1, \hat{u}_2)$ in order to obtain an appropriate control steering our system from an initial state to a preassigned final state. Under some general geometric assumptions for S_1, theoretical results presented in Section 4.3 guarantee that, whatever the starting control we choose, Eq. (4.12) is complete and provides the correct control law at the end of the integration. We use the classical Euler scheme to integrate Eq. (4.12). Note that Theorem 1 in [25] ensures that, once there exists a global solution to Eq. (4.12), then for any "reasonable" Galerkin approximation of the control space and "reasonable" numerical scheme for the derivatives, there exists a global solution for the corresponding numerical approximation of Eq. (4.12).

In the following two paragraphs, we give details about the two key points for the numerical implementation which are the evaluation of a right inverse of $D\phi_p(u)$ and the integration of Eq. (4.15).

4.4.2 Computing $D\phi_p(u)$

We first need to define a field of covectors along $\gamma_{p,u}$. For $z \in T^*_{\phi_p(u)}M$, let $\lambda_{z,u} : [0,1] \to T^*M$ be the field of covectors along $\gamma_{p,u}$ satisfying (in coordinates) the adjoint equation along

$\gamma_{p,u}$ with terminal condition z, *i.e.*, $\lambda_{z,u}$ is a.c., $\lambda_{z,u}(1) = z$ and for a.e. $t \in [0,1]$,

$$\dot{\lambda}_{z,u}(t) = -\lambda_{z,u}(t) \cdot \left(\sum_{i=1}^{2} u_i(t) DF_i(\lambda_{z,u}(t)) \right). \tag{4.17}$$

If X is a smoothvector field over M, the switching function $\varphi_{X,z,u}(t)$ associated to X is the evaluation of $\lambda \cdot X(x)$, the Hamiltonian function of X along $(\gamma_{p,u}, \lambda_{z,u})$, *i.e.*, for $t \in [0,1]$,

$$\varphi_{X,z,u}(t) := \lambda_{z,u}(t) \cdot X(\gamma_{p,u}(t)),$$

(see for instance [25] for more details). Then $D\phi_p(u)$ can be computed as follows : for $z \in T^*_{\phi_p(u)}M$ and $u, v \in H$,

$$z \cdot D\phi_p(u)(v) = (v, \varphi_{z,u})_H, \tag{4.18}$$

where the switching function vector $\varphi_{z,u}(t)$ is the solution of the following Cauchy problem, defined (in coordinates) below, by (cf. [23])

$$\begin{array}{rcl} \dot{\varphi}_1 &=& -u_2 K \varphi_3, \\ \dot{\varphi}_2 &=& u_1 K \varphi_3, \\ \dot{\varphi}_3 &=& -u_2 \varphi_4 + u_1 \varphi_5, \\ \dot{\varphi}_4 &=& -u_2 K \varphi_3, \\ \dot{\varphi}_5 &=& u_1 K \varphi_3. \end{array} \tag{4.19}$$

with terminal condition $\varphi_{z,u}(1) = z$. The reader is referred to [73, Sections 5.2.2 and 5.2.3] for a detailed computation of the differential of the end-point map in general.

In practice, since the discrete $D\phi_p(u)$ is a 5×5 matrix and its image is given by Eq. (4.18), it suffices to take five independents vectors in $\mathbb{R}^5$ as final conditions z, for instance the five elements in the canonical basis of $\mathbb{R}^5$ and integrate Eq. (4.19) in reverse time.

In our simulations, a fourth-order Runge-Kutta numerical scheme is used for integration, the scalar product $(\cdot,\cdot)_H$ in control space H is evaluated by Gaussian quadrature and the Gaussian curvature K is computed by using the following proposition, cf. [13].

Proposition 4.3. *Let S be (a bounded connected component of) the zero-set of $f : \mathbb{R}^3 \to \mathbb{R}$, and define a, b, c by*

$$\det \begin{pmatrix} \nabla^2 f - \lambda I_3 & \nabla f \\ (\nabla f)^T & 0 \end{pmatrix} = a + b\lambda + c\lambda^2, \tag{4.20}$$

where $\nabla^2 f$ is the matrix of the second derivatives of f and I_3 the identity 3×3 matrix.

With this notation, one has

$$K = \frac{a/c}{\|\nabla f\|^2}. \tag{4.21}$$

Explicit computations show that $c = -\|\nabla f\|^2$ and

$$a = \det \begin{pmatrix} \nabla^2 f & \nabla f \\ (\nabla f)^T & 0 \end{pmatrix}.$$

Hence, we have

$$K = -\frac{\det \begin{pmatrix} \nabla^2 f & \nabla f \\ (\nabla f)^T & 0 \end{pmatrix}}{\|\nabla f\|^4}. \tag{4.22}$$

The gradient vector ∇f is then evaluated by a classical right-shifting finite difference scheme, and $\nabla^2 f$ by a centered one. For example, if $X = (x, y, z)$, then $f_x(X)$ is given by

$$\frac{f(x+\varepsilon, y, z) - f(x, y, z)}{\varepsilon}, \tag{4.23}$$

and $f_{xx}(X)$ by

$$\frac{f(x+\varepsilon, y, z) - 2f(x, y, z) + f(x-\varepsilon, y, z)}{\varepsilon^2}, \tag{4.24}$$

with $\varepsilon > 0$ small enough.

4.4.3 Lifting the plane curve $\hat{c}_2$ on S_1

Note that the curvature K appearing in Eq. (4.19) is taken at the final contact point on the surface S_1 after it has rolled along the piecewise constant curve $\hat{c}_2$. Thus, in order to locate the final point, we need to "lift" the plane curve $\hat{c}_2$ on S_1, and the lifting dynamics are given by Eq. (4.15). However, since the geodesic coordinates involved in Eq. (4.15) are not given explicitly in practice, our numerical *lifting* method is based on Proposition 4.1. See also Remark 4.2.

On each interval $[t_i, t_{i+1}]$, the approximate control curve $\hat{c}_2$ is a straight line (i.e. a geodesic in $\mathbb{R}^2$), and then, by Proposition 4.1, the lifting curve $\hat{c}_1$ on S_1 is also a geodesic on each interval $[t_i, t_{i+1}]$ for all $i = 0, \dots, N-1$. Then, from the initial contact point X_0 on S_1, we can integrate successively the geodesic equation on each $[t_i, t_{i+1}]$ with initial conditions equal to $\hat{c}_1(t_i)$ and $(\hat{u}_1^i, \hat{u}_2^i)$, for $i = 0, \dots, N-1$.

Let us write explicitly the geodesic equation to be integrated (see for instance [13] for more details). Recall that a curve $c : [0, 1] \to S_1$ is a geodesic curve if it verifies Eq. (4.1). In the case where S_1 is an immersed surface in $\mathbb{R}^3$, Eq. (4.1) is equivalent to

$$\ddot{c}(t) \perp T_{c(t)} S_1, \tag{4.25}$$

for almost all t in $[0, 1]$.

When S_1 is defined as (a bounded connected component of) the zero-level set of a real-valued function $f : \mathbb{R}^3 \to \mathbb{R}$, we have $\nabla f(x) \perp T_x S_1$ at every $x \in S_1$. Thus, Eq. (4.25) becomes

$$\ddot{c} = \left\langle \ddot{c}, \frac{\nabla f(c)}{\|\nabla f(c)\|} \right\rangle \frac{\nabla f(c)}{\|\nabla f(c)\|}, \tag{4.26}$$

where $\langle , \rangle$ is the scalar product in $\mathbb{R}^3$.

Furthermore, since c is a curve traced on S_1, we also have

$$\langle \dot{c}(t), \nabla f(c(t)) \rangle = 0, \tag{4.27}$$

for almost all t in $[0, 1]$. Then, by deriving Eq. (4.27) with respect to t, we get

$$\langle \ddot{c}, \nabla f(c) \rangle + \langle \dot{c}, \nabla^2 f(c) \dot{c} \rangle = 0. \tag{4.28}$$

Finally, summing up Eq. (4.26) and Eq. (4.28) together, we get

$$\ddot{c} = -\frac{\dot{c}^T \nabla^2 f(c) \dot{c}}{\|\nabla f(c)\|^2} \nabla f(c). \tag{4.29}$$

We use again a fourth-order Runge-Kutta scheme for numerical integration of Eq. (4.29).

An additional difficulty is that the numerical integration is not performed in an Euclidean space, but on a manifold S_1. Assume that we are at point $x \in S_1$ at time t. Then, at time $t + \delta t$, we move to $X_{\text{new}} = X + (\delta t)d$ with $d \in T_x S_1$, but X_{new} does not belong to S_1 if d is nonzero. Therefore, at each integration step, we have to "project" X_{new} on S_1.

More precisely, assume that the point $(0, 0, 0)$ is inside the convex body S_1. Since S_1 is defined as (a bounded connected component of) the zero-level set of a smooth function f, we assume that $|f(X_{\text{new}})| \leq \varepsilon$ for some $\varepsilon << 1$, i.e X_{new} is close to S_1. Then, there exists a unique real number μ close to 1 such that $f(\mu X_{\text{new}}) = 0$, as a simple consequence of the convexity of S_1. The "projection" issue to be addressed is clearly a local one and therefore, Newton's method is efficient for finding μ. The derivative with respect to μ is also needed, it is evaluated by a finite difference scheme similar to Eq. (4.23).

4.5 Simulations

We have applied the numerical continuation method presented above for motion planning problem of several bodies rolling on the Euclidean plane. We first present the rolling of a flattened ball and an egg. We then give simulation in a case where the rolling body does not have a symmetry of revolution. Still, the CM works quite efficiently.

Let us point out that we have written a Matlab program which provided us with the figures presented below. In particular, these figures contain buttons and windows of the Matlab interface. For the convenience of the reader, we will recall all the equations defining the rolling surfaces in the corresponding paragraphs. All the figures show the starting and ending contact points and orientations in the top left, the current trajectory on the plane together with the starting and ending configurations of the rolling body in the top right, and the corresponding trajectory on the body in the bottom left. Since the key point is to show how the continuation method modifies smoothly an arbitrary non singular plane curve to achieve a "right" one, for each test case, we show in the first figure the initial curve that we have chosen, in the second and third figures two intermediate phases adjusting the contact point and orientation, in the fourth figure, the final curve computed by the algorithm as well as the body rolling along this curve, and finally, in the last figure, the matching between the real final state and the preassigned one.

We also note that the computation time is on average 30 seconds (2.2 GHz Intel Core 2 Duo, 1.6 G memory) for 70 iterations with $N = 100$ for the discretization of control space H.

4.5.1 Flattened ball rolling on the plane

This flattened ball is defined by the zero-level set of the function

$$f(x, y, z) = x^2 + y^2 + 5z^2 - 1. \tag{4.30}$$

The gradient $\nabla f(x, y, z)$ is equal to $(2x, 2y, 10z)^T$. One can check that it is never equal to zero on the zero-level set of Eq. (4.30). Then Eq. (4.22) and Eq. (4.29) are always well defined.

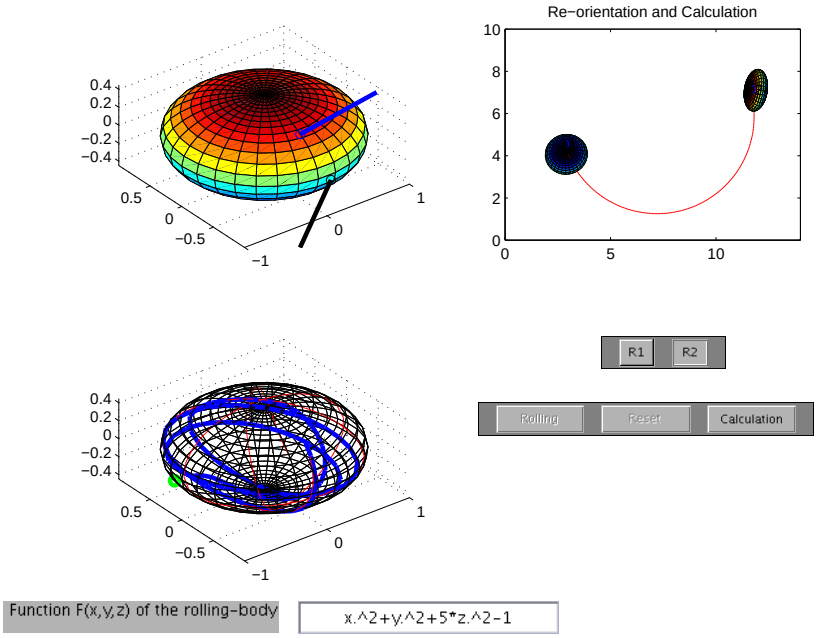


FIGURE 4.1 – Initial and final positions of contact point and orientations of the flattened ball ($s = 0$).

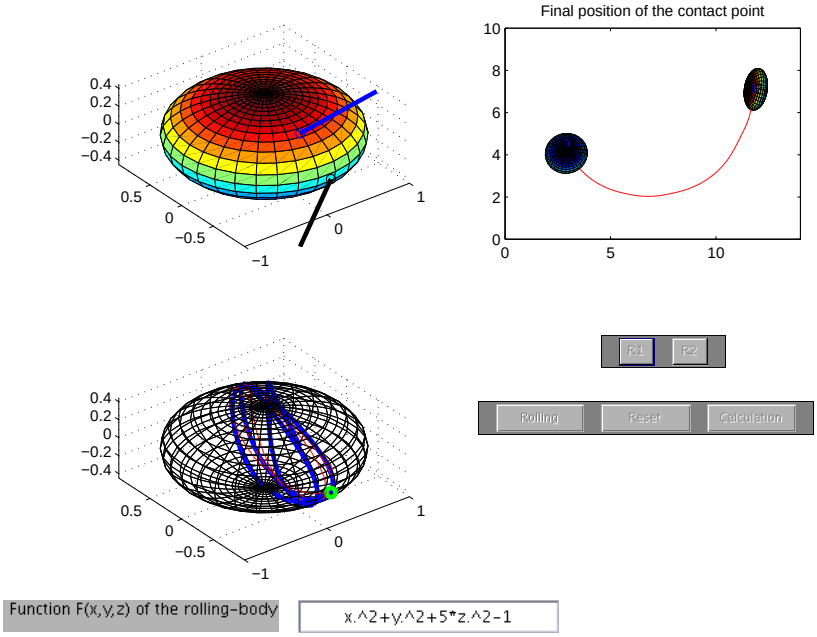


FIGURE 4.2 – Computation for adjusting the final position of contact point by continuation method ($s = 35$).

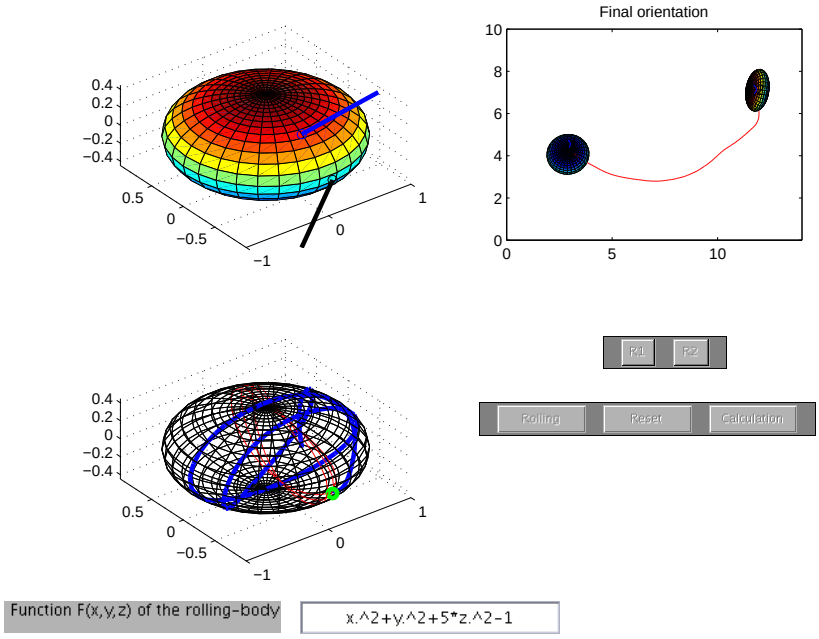


FIGURE 4.3 – Computation for adjusting the final orientation of the flattened ball by continuation method ($s = 70$).

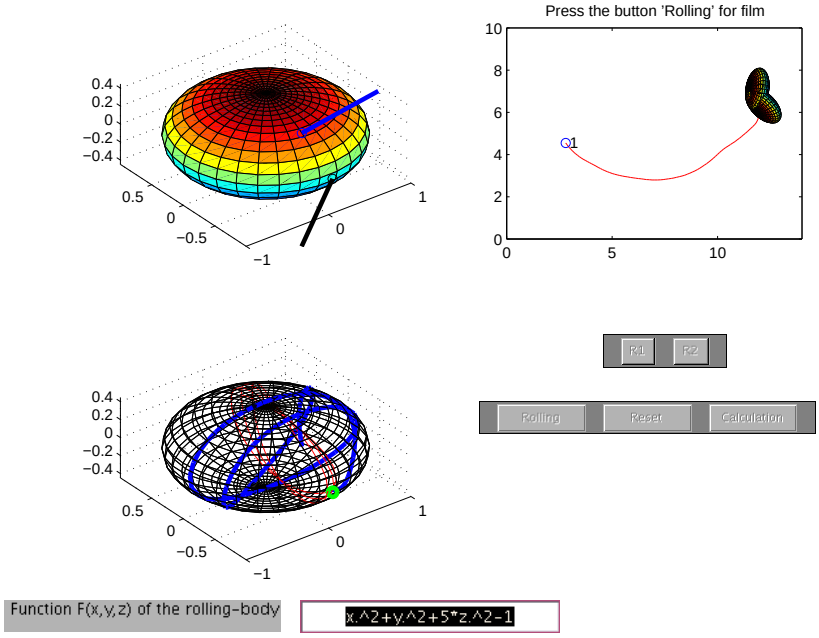


FIGURE 4.4 – Flattened ball rolling along the curve before reaching the final position.

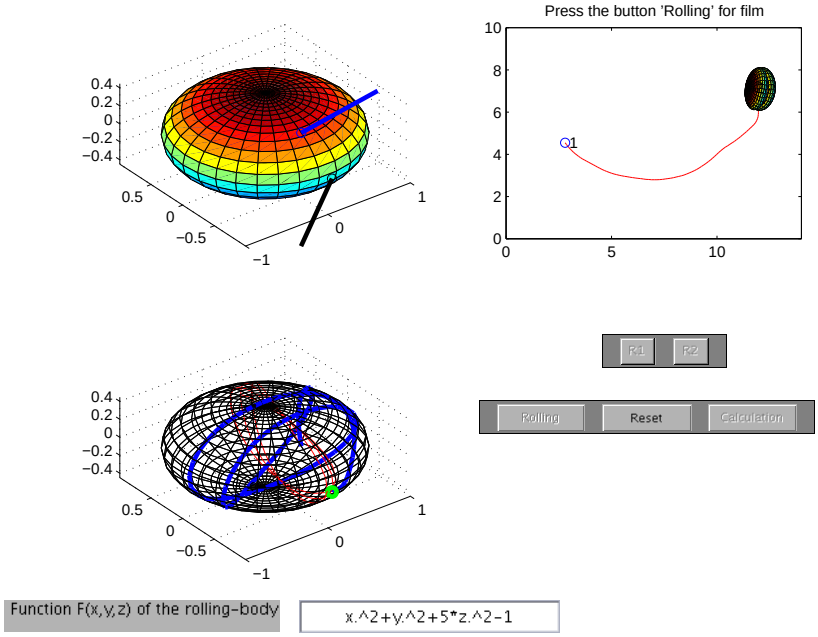


FIGURE 4.5 – Matching at the end of rolling.

4.5.2 Egg rolling on the plane

This "egg" is defined by one bounded connected component of the zero-level set of the function

$$f(x,y,z) = \frac{x^2+y^2}{1-0.4z} + \frac{z^2}{4} - 1. \tag{4.31}$$

We note that $\nabla f(x,y,z) = (\frac{2x}{1-0.4z}, \frac{2y}{1-0.4z}, \frac{0.4(x^2+y^2)}{(1-0.4z)^2} + \frac{z}{2})^T$. One can check that it is never equal to zero on the zero-level set of Eq. (4.31) and therefore Eq. (4.22) and Eq. (4.29) are always well defined.

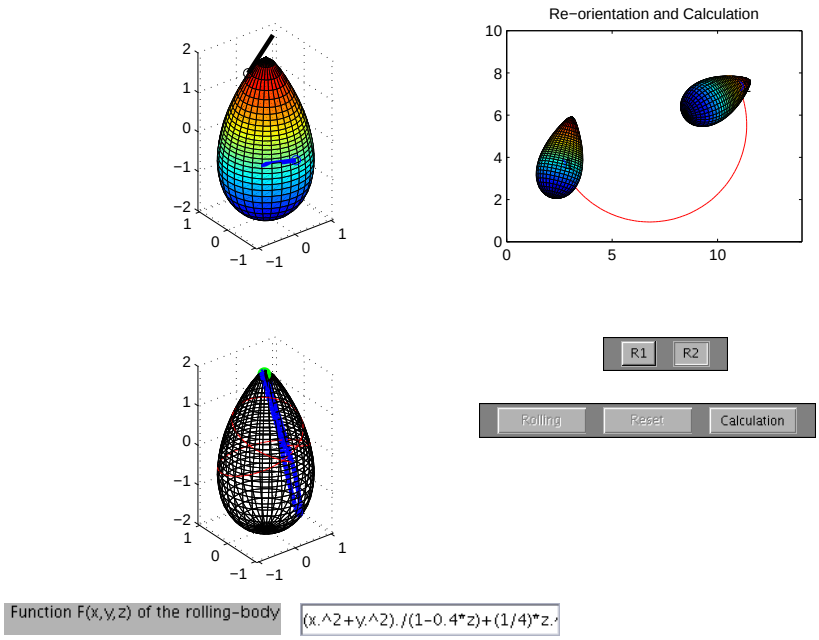


FIGURE 4.6 – Initial and final positions of contact point and orientations of the egg ($s = 0$).

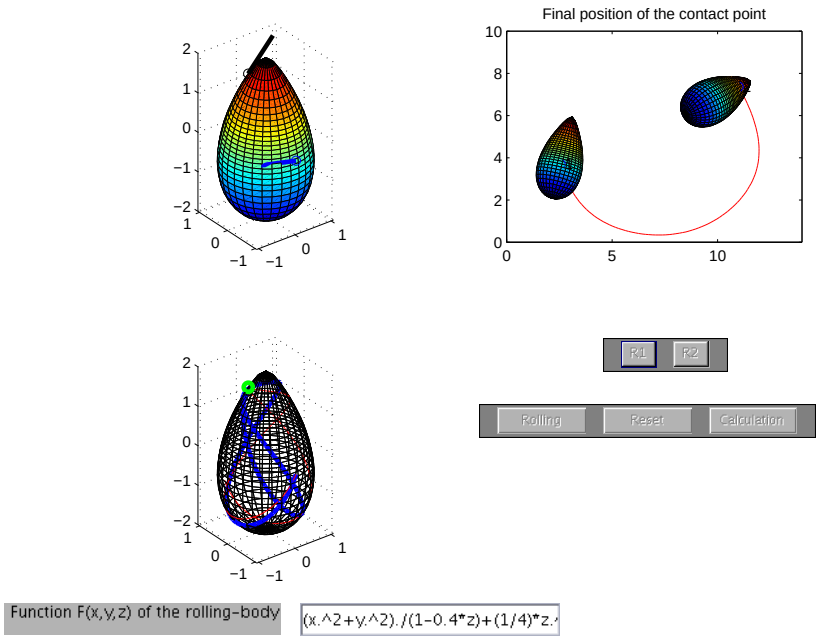


FIGURE 4.7 – Computation for adjusting the final position of contact point by continuation method ($s = 35$).

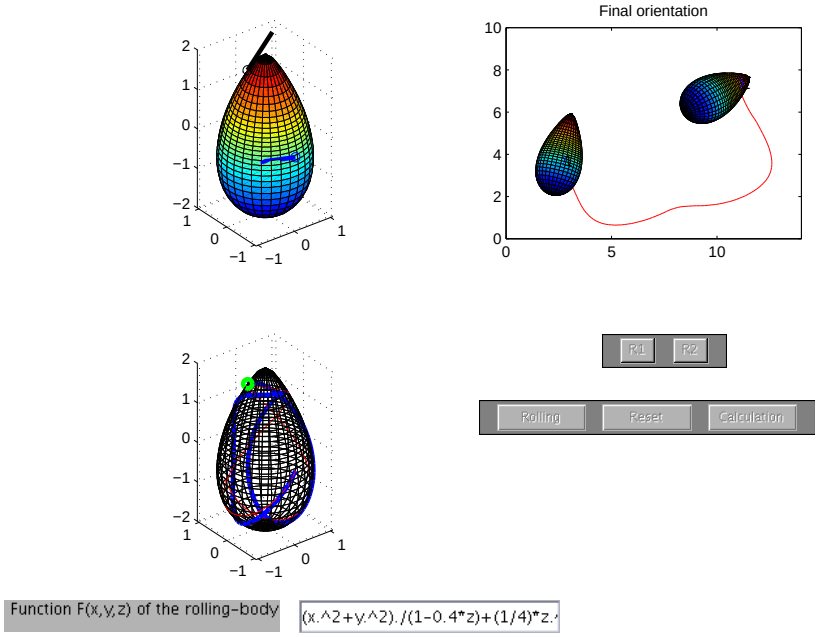


FIGURE 4.8 – Computation for adjusting the final orientation of the egg by continuation method ($s = 70$).

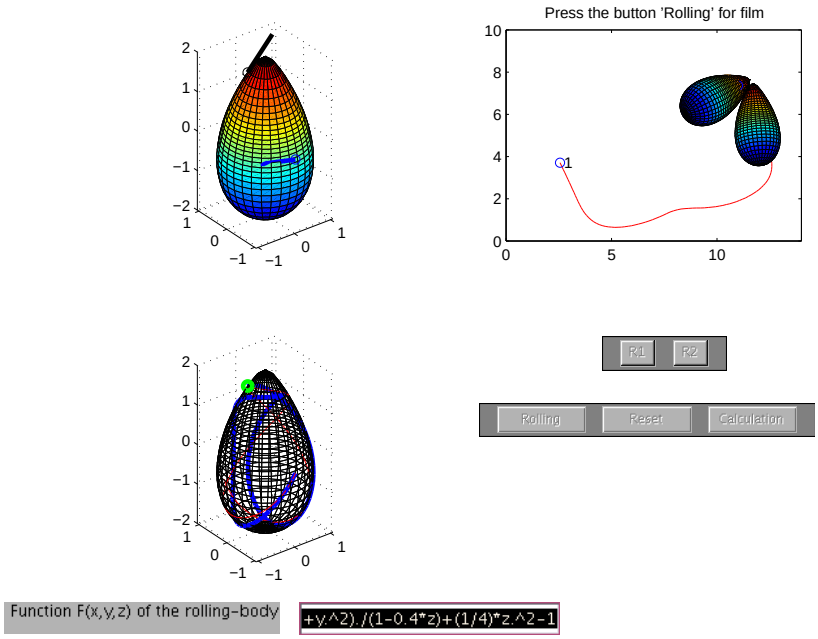


FIGURE 4.9 – Egg rolling along the curve before reaching the final position.

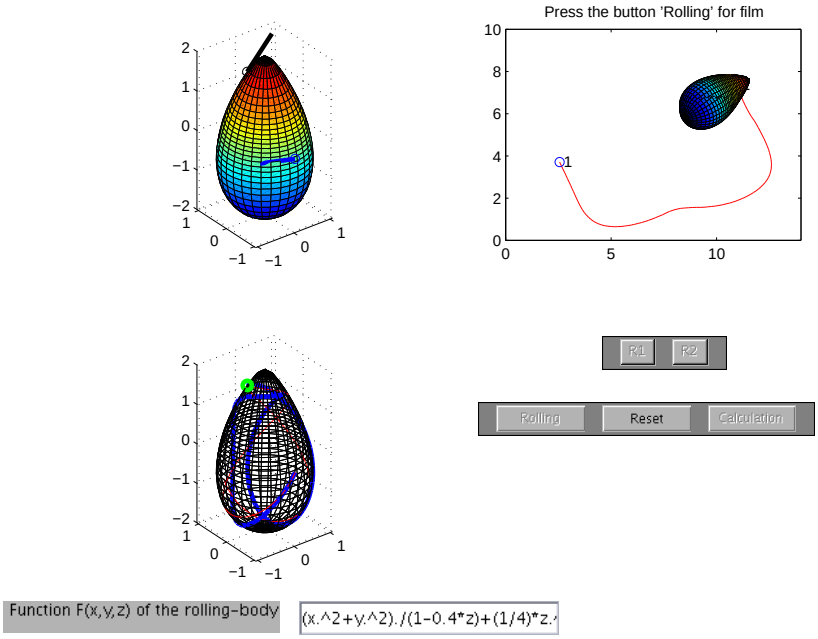


FIGURE 4.10 – Matching at the end of rolling.

4.5.3 More general case

In Section 4.3, the global convergence of continuation method has been proven for rolling of convex body with symmetric axis. However, we show in the subsequent simulations that the continuation method still works numerically in more general cases, even though a theoretical convergence result is not available. This illustrates the robustness of the method.

For example, we take the convex body without symmetric axis, defined by one bounded connected component of the zero-level set of the function

$$f(x,y,z) = \frac{x^2}{1-0.5y} + \frac{2y^2}{1-0.1z} + \frac{0.5z^2}{1-0.3x-0.1y} - 1. \tag{4.32}$$

We note that

$$\nabla f(x,y,z) = \begin{pmatrix} \frac{2x}{1-0.5y} + \frac{0.3}{(1-0.3x-0.1y)^2} \\ \frac{4y}{1-0.1z} + \frac{0.1}{(1-0.3x-0.1y)^2} \\ \frac{0.2y^2}{(1-0.1z)^2} + \frac{z}{1-0.3x-0.1y} \end{pmatrix}.$$

One can check that it is never equal to zero on the zero-level set of Eq. (4.32) and therefore Eq. (4.22) and Eq. (4.29) are always well defined.

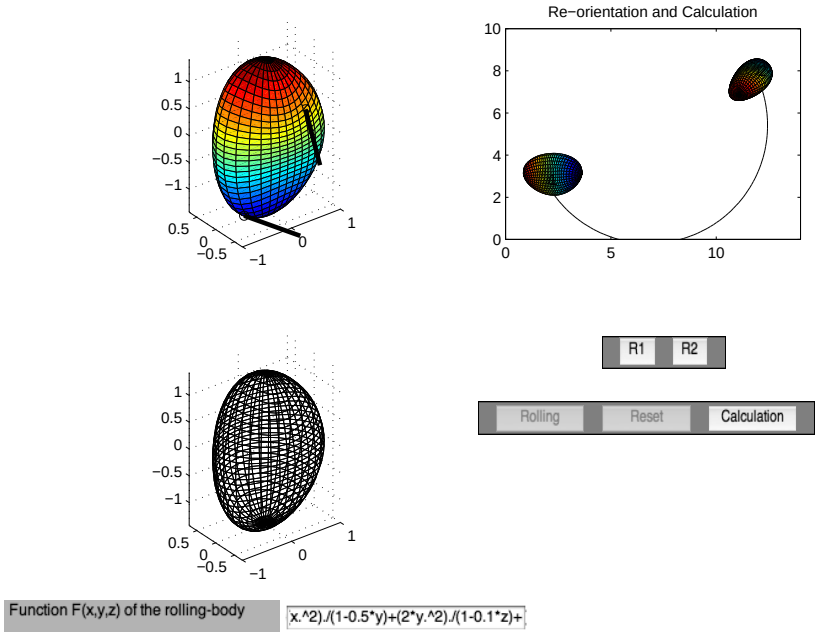


FIGURE 4.11 – Initial and final positions of contact point and orientations of the convex body ($s = 0$).

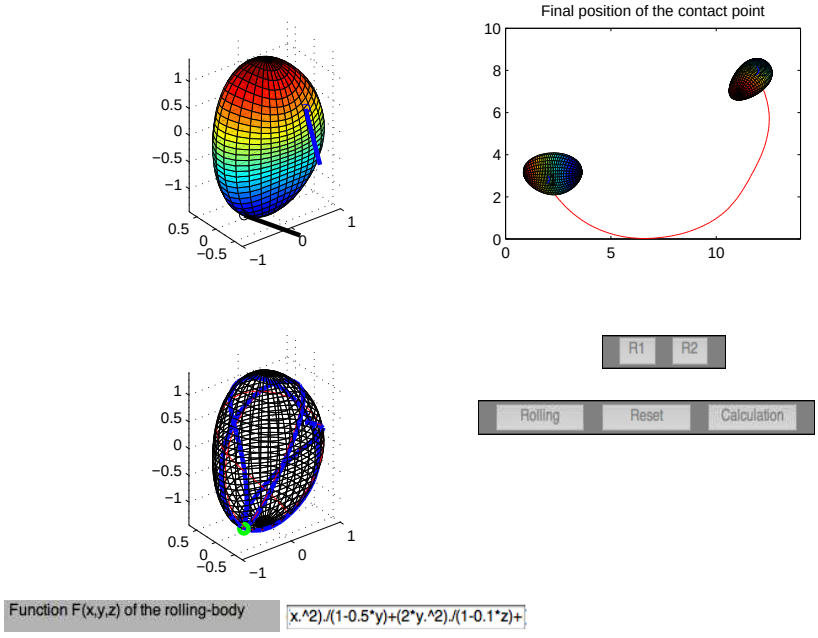


FIGURE 4.12 – Computation for adjusting the final position of contact point by continuation method ($s = 35$).

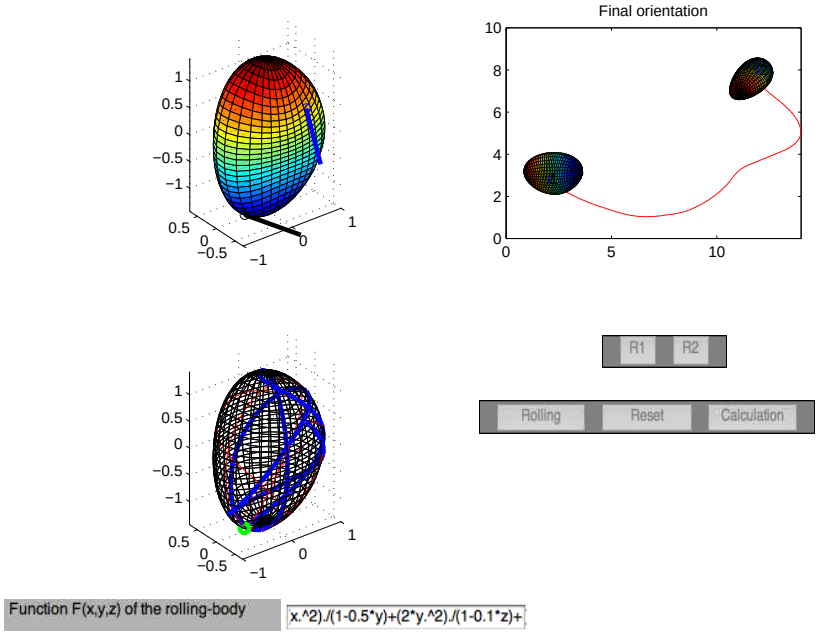


FIGURE 4.13 – Computation for adjusting the final orientation of the convex body by continuation method ($s = 70$).

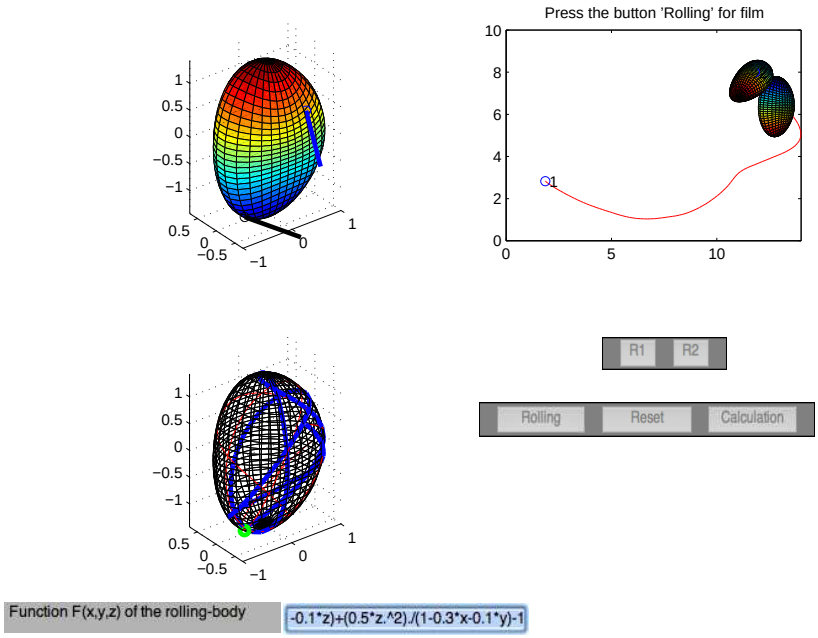


FIGURE 4.14 – Convex body rolling along the curve before reaching the final position.

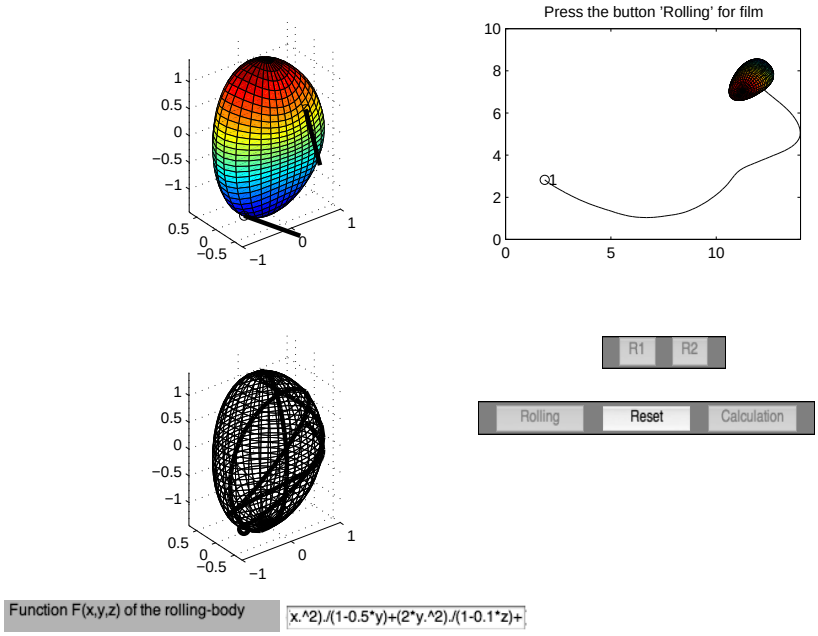


FIGURE 4.15 – Matching at the end of rolling.

4.6 Discussion and Conclusion

The main difficulty in the motion planning for convex bodies with rolling constraints relies on the fact that the displacement and the change of orientation cannot be dissociated one from the other. In the case of Problem **L1.**, every closed curve on S_1 can be associated with an element of the group $SE(2)$, and the concatenation of two closed curves corresponds to the group operation for $SE(2)$. This correspondence was implicitly mentioned in [66] via the Gauss-Bonnet Theorem, and it was explicitly and systematically explored in [15] for the construction of the lattice structure and basic actions. However, using this point of view, exact computations as presented in [66] cannot be extended beyond the plate-ball system, and approximate computations based on some discretization of the state space presented in [15] produce highly oscillating trajectories as they are obtained by concatenating a large number of basic actions composed of rolling along some closed curves defined on S_1.

In this paper, we have adopted a more global point of view which is to modify continuously, via the continuation method developed in [23, 25], an arbitrary non singular control (any plane curve which is not straight line in the case of problem **L1**) in order to achieve one control which steers the system from a given initial state to preassigned final state. We have implemented this method to solve the problem **L1-1** (rolling of general strictly convex bodies on the free plane). We have shown through several examples the robustness and the convergence speed of this method. It is worth pointing out that the only knowledge about the surface S_1 required by the numerical implementation of our method is the Gaussian curvature function K_1 of S_1. We have assumed in Section 4.4 that there exists a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $S_1 = f^{-1}(0)$, then K_1 can be directly expressed (and numerically evaluated) from f. The numerical advantage of this *level-set* approach relies on the fact that our motion planning algorithm can be implemented without dealing with any change of local parameterization (chart) of S_1. We also note that this assumption is not restrictive. Indeed, for any compact

convex body S_1, if we assume that the origin 0 is inside of S_1, then we can define f as follows :

$$f(x) = t - 1, \quad \text{if } \frac{x}{t} \in S_1, \ t > 0. \tag{4.33}$$

Then, $S_1 = f^{-1}(0)$. Moreover, one can show that f is convex. Therefore, the Gaussian curvature can be computed from this function f since any continuous convex function admits second derivatives almost everywhere (cf. [47, Chap. I, Sect. 5]).

Our method can be adapted in order to solve **L1-2** (a convex body S_1 rolling on a plane with obstacles on the latter) by potential fields and this is the purpose of a forthcoming paper. Solving **L1-3** (strictly convex body S_1 rolling on a plane with prohibited regions on S_1) is more challenging since we must deal numerically with local charts of S_1. A possible way to address this issue is to use penalization techniques. An additional difficulty regarding **L2** (one strictly convex body rolling on the top of the other without or with prohibited regions) relies on the fact that the invertibility of $D\phi_p$ involved in the Path Lifting Equation (4.12) as well as the non-explosion condition require $K_2 - K_1 \neq 0$ at the contact point, but this condition may not be globally verified for two general smooth convex bodies.

4.7 Appendix : Continuation method applied to the rolling-body problem

For the sake of completeness, we summarize in this Appendix principal results regarding the **Condition 4.1** in the case of a strictly convex surface S_1 rolling on a plane. The reader can refer to [23] and references therein for a complete discussion on this issue. Roughly speaking, it is shown in [23] that if S_1 verifies a simple geometric property (see **Condition 4.2** below), then there exists a compact subset $\mathcal{K}$ in the state space M verifying the **Condition 4.1**, which is large enough to completely resolve the MPP.

The existence of a large compact $\mathcal{K}$ verifying the **Condition 4.1** requires a "small" singular set $\phi_p(S_p)$ characterized by Proposition 4.2. This condition is guaranteed by the existence of a periodic geodesic on S_1, stable for the geodesic flow of S_1. More precisely, let d_1 be the distance function associated to the Riemannian metric of S_1 induced by the usual metric of $\mathbb{R}^3$.

Condition 4.2. *We say that a surface S_1 verifies* **Condition 4.2** *if there exists a geodesic curve $\gamma : \mathbb{R}^+ \to T_1S_1$, $L > 0$ and $\rho_0 > 0$ such that*

(s) $\gamma(t + L) = \gamma(t)$ for all $t \geq 0$ (cf. [58]) *;*

(p) $\forall \rho < \rho_0$, $\exists \eta(\rho) > 0$, $\forall y_0 \in N_\rho(G)$, $\forall t \geq 0$, we have

$$\phi(y_0, t) \in N_\eta(G),$$

and

$$lim_{\rho \to 0} \eta(\rho) = 0,$$

where $G := \gamma([0, L])$, $N_\rho(G)$ is the open set of points $y \in T_1S_1$ such $d_1(y, G) < \rho$ and $\phi(y, t)$ is the geodesic flow of T_1S_1.

It is shown in [58] that **Condition 4.2** holds true for any convex compact surface having a symmetry of revolution and it is generic within the convex compact surfaces verifying $K_{\min}/K_{\max} > \frac{1}{4}$, where $K_{\min}$ and $K_{\max}$ denote the minimum and the maximum respectively of the Gaussian curvature over the surface.

Assume now that S_1 verifies **Condition 4.2** and let G be the support of the periodic geodesic. Then, $\rho \in (0, \rho_0)$, define $\mathcal{K}_{\bar{\rho}}$ as the set complement in T_1S_1 of $N_\rho(G) \times L$, where L is a fixed line in $\mathbb{R}^2$. The next proposition, proved in [23], tackles the non-explosion issue relative to the global existence of the solution of the Path Lifting Equation.

Proposition 4.4. *There exists a line $L \in \mathbb{R}^2$ and $\bar{\rho} > 0$ such that the corresponding $\mathcal{K}_{\bar{\rho}}$ verifies* **Condition 4.1**.

Then, we have the following proposition guaranteeing that the continuation method can be successfully applied for solving the rolling-body motion planning problem.

Proposition 4.5. *With above notations, for every path $\pi : [0,1] \to \mathcal{K}_{\bar{\rho}}$ of class C^1 and every control $\bar{u} \in H$ such that $\pi(0) = \phi_p(\bar{u})$, the solution of the Path Lifting Equation (4.12), with initial condition equal to $\bar{u}$, exists globally over $[0,1]$.*

We now describe how Proposition 4.5 can be applied to the rolling-body motion planning problem. Assume that one wants to roll the body from an initial position $p \in M$ to a final one $q \in M$.

Let us first assume that both p and q belong to $\mathcal{K}_{\bar{\rho}}$. We note that, since γ is periodic, $N_\rho(G)$ is diffeomorphic to the product of a small two-dimensional ball and a closed path on S_1. Therefore, $\mathcal{K}_{\bar{\rho}}$ is closed and arc-connected. We begin by taking an arbitrary control $\bar{u}$ which does not belong to S_p. Then we choose a C^1-path $\pi : [0,1] \to \mathcal{K}_{\bar{\rho}}$ such that $\pi(0) := \phi_p(\bar{u})$ and $\pi(1) := q$. Proposition 4.5 guarantees that, by integrating Eq. (4.12) over $[0,1]$ with initial condition equal to $\bar{u}$, we obtain a curve $\Pi : [0,1] \to H$ such that $\phi_p(\Pi(s)) = \pi(s)$ for $s \in [0,1]$. In particular, we have $\phi_p(\Pi(1)) = \pi(1) = q$, which means that the control $u := \Pi(1)$ solves the motion planning problem. If, for instance, p does not belong to $\mathcal{K}_{\bar{\rho}}$, it suffices first to roll the body along one geodesic which brings it to a point $\tilde{p}$ belonging to $\mathcal{K}_{\bar{\rho}}$, then we consider $\tilde{p}$ as the new initial condition, and the continuation method will apply. We recall that geodesic curves are admissible trajectories for the rolling body problem by Proposition 4.1.

Deuxième partie

Etude de la contrôlabilité spectrale des équations de Schrödinger linéarisées

Chapitre 5

Spectral controllability for 2D and 3D linear Schrödinger equations

Le contenu de ce chapitre fait l'objet d'un article en collaboration avec K. Beauchard, Y. Chitour et D. Kateb, publié dans le *Journal of Functional Analysis*, volume 256 (2009), numéro 12 (juin), pp. 3916–3976.

Sommaire

5.1 Introduction

Let us consider a quantum particle in an infinite square potential well of $\mathbb{R}^n$, $n \in \{1, 2, 3\}$ subjected to a uniform (in space) time dependent electric field $u : t \mapsto u(t) \in \mathbb{R}^n$. Let Ω be the domain of $\mathbb{R}^n$ corresponding to the bottom of the well. This physical system is modeled by a wave function

$$\begin{array}{rccc} \psi : & \mathbb{R}_+ \times \Omega & \to & \mathbb{C} \\ & (t, q) & \mapsto & \psi(t, q), \end{array}$$

such that $|\psi(t,q)|^2 dq$ represents the probability of the particle to be in the volume dq surrounding the point q at time t. Thus, the wave function ψ lives on the $L^2(\Omega,\mathbb{C})$-sphere S as it is well known that the $L^2(\Omega,\mathbb{C})$-norm of the wave function ψ is preserved over time . Under the dipolar moment approximation, this wave function solves the following Schrödinger equation

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,q) = -\Delta\psi(t,q) - \langle u(t),\mu(q)\rangle\psi(t,q), & (t,q)\in\mathbb{R}_+\times\Omega, \\ \psi(t,q) = 0, & (t,q)\in\mathbb{R}_+\times\partial\Omega, \end{cases} \tag{5.1}$$

where $\mu\in C^0(\overline{\Omega},\mathbb{R}^n)$ is the dipolar moment and $\langle .,.\rangle$ denotes the usual scalar product on $\mathbb{R}^n$. The system (5.1) is a non linear control system in which

– the state is the wave function ψ with $\psi(t)\in S$, for every $t\geq 0$,
– the control is the electric field $u: t\in\mathbb{R}_+\mapsto u(t)\in\mathbb{R}^n$.

Studying controllability properties of the control system (5.1) reveals interesting features. For instance, Turinici proved in [105] that, the system (5.1) is not controllable in $H^2\cap H_0^1(\Omega,\mathbb{C})$ with controls u in $L^r_{loc}(\mathbb{R}_+,\mathbb{R}^n)$, $r\in(1,+\infty)$. This result is a corollary of a more general result about the controllability of bilinear control systems, due to Ball, Marsden and Slemrod in [8]. However, it has been proved in [9] that the system (5.1) in 1D, with $\Omega=(-1/2,1/2)$ and $\mu(q)=q$ is locally controllable around the ground state in $H^7((-1/2,1/2),\mathbb{C})$ with $H_0^1((0,T),\mathbb{R})$ controls, when T is large enough. This system is even controllable between eigenstates, as proved in [10]. Therefore the non controllability result emphasized in [105] is essentially due to a choice of functional spaces that do not allow the controllability, but this controllability holds in other satisfying functional spaces. At the moment, in 2D or 3D, no positive exact controllability result is known for (5.1).

We can also consider a similar non linear system. The quantum particle is now placed in a moving infinite square potential well of $\mathbb{R}^n$, $n\in\{1,2,3\}$. Let Ω be the domain of $\mathbb{R}^n$ corresponding to the bottom of the well. It is proved by Rouchon in [85] that this physical system is represented by the following Schrödinger equation

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,q) = -\Delta\psi(t,q) - \langle u(t),\mu(q)\rangle\psi(t,q), & (t,q)\in\mathbb{R}_+\times\Omega, \\ \psi(t,q) = 0, & (t,q)\in\mathbb{R}_+\times\partial\Omega, \\ \dot{d}(t) = s(t), \\ \dot{s}(t) = u(t), \end{cases} \tag{5.2}$$

where ψ is the wave function of the particle in the moving frame, $u:=\ddot{d}$ is the acceleration of the well, s is the speed of the well, d is the position of the well and $\mu(q)=q$ (but in this article, we will study this system for more general functions μ). The system (5.2) is a nonlinear control system with state, the triple (ψ,s,d) with $\psi(t)\in S$, for every $t\geq 0$, and control, the acceleration of the well $u: t\in\mathbb{R}_+\mapsto\mathbb{R}^n$. In 1D, with $\Omega=(-1/2,1/2)$, the local controllability around the eigenstates and the controllability between eigenstates of (5.2) is proved in [10].

A classical approach to prove the local controllability of non linear systems such as (5.1) and (5.2) around a reference trajectory consists in proving first, the controllability of the linearized system around the reference trajectory and second, the local controllability of the non linear system around the reference trajectory, with the help of an inverse mapping theorem. If the linearized system around the reference trajectory is not controllable, one may use the return method advocated by Coron (cf. [33, 32] and references therein, and [9], [10] for applications to 1D Schrödinger equations). This method relies on the study of another reference trajectory of the non linear system admitting a controllable linearized system.

Therefore, it is natural to linearize (5.1) and (5.2) along "simple" trajectories, for instance along the one corresponding to the zero control, $u\equiv 0$ and to study the controllability of

the resulting linear system. For $k \in \mathbb{N}^*$, the *eigenstate* $\psi_k(t,q) := \phi_k(q)e^{-i\lambda_k t}$ defines such a trajectory ($(\psi = \psi_k, u \equiv 0)$ for (5.1) and $(\psi = \psi_k, s \equiv 0, d \equiv 0, u \equiv 0)$ for (5.2)), where $(\phi_k)_{k\in\mathbb{N}^*}$ is a complete orthonormal system of eigenfunctions for $-\Delta_\Omega^D$, the Laplacian operator on Ω with Dirichlet boundary condition, and $(\lambda_k)_{k\in\mathbb{N}^*}$ are the corresponding non-decreasing sequence of eigenvalues counted with their multiplicity. In the particular case $k = 1$, ψ_1 is called the *ground state* and the following systems are the linearized systems respectively of (5.1) around the ground state,

$$\begin{cases} i\frac{\partial\Psi}{\partial t}(t,q) = -\Delta\Psi(t,q) - \langle v(t), \mu(q)\rangle\psi_1(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{5.3}$$

and of (5.2) around the trajectory $((\psi = \psi_1, s \equiv 0, d \equiv 0), u \equiv 0)$,

$$\begin{cases} i\frac{\partial\Psi}{\partial t}(t,q) = -\Delta\Psi(t,q) - \langle v(t), \mu(q)\rangle\psi_1(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \\ \dot{d}(t) = s(t), \\ \dot{s}(t) = v(t). \end{cases} \tag{5.4}$$

In this paper, we only study controllability properties of systems (5.3) and (5.4).

Let us recall classical results about the controllability of these two systems in 1D, results being the starting point of the strategies developed in [9] and [10] for the nonlinear systems (5.1) and (5.2). Their proof will be sketched in Section 5.2 in order to explain the difficulties arising in their generalization to the 2D and 3D cases. For system (5.3), $\Omega = (0,1)$ and, if s is a non negative real number, let $H^s_{(0)}((0,1),\mathbb{C})$ be equal to $D(A^{s/2})$ where $D(A) := H^2 \cap H^1_0((0,1),\mathbb{C})$ and $A\varphi := -\varphi''$. Then, up to a condition satisfied by the dipolar moment μ (see Proposition 5.2 for a detailed statement), the system (5.3) is controllable in $H^3_{(0)}((0,1),\mathbb{C})$ with control functions in $L^2((0,T),\mathbb{R})$ for every $T > 0$. As regards controllability for system (5.4), we show that it is not exact controllable in finite time for the 1D problem and we describe the reachable set. The crucial technical reason for that lies in the fact that the eigenvalues of Δ_Ω^D verify a uniform gap condition, i.e. there exists $\rho > 0$ such that, for every positive integer, we have $\lambda_{k+1} - \lambda_k \geq \rho$. However, in 2D, the existence of a regular domain Ω of $\mathbb{R}^2$ such that the eigenvalues of Δ_Ω^D present a uniform gap is still an open problem and in 3D, no uniform gap is possible because of the Weyl formula. Therefore, exact controllability of (5.3) and (5.4) in 2D and 3D is not a trivial question and it is thus natural to study a weaker controllability property for this system. This is why we investigate, in this article, the *spectral controllability* of systems (5.3) and (5.4). To define that concept of controllability, let us denote $\mathcal{D}$, the linear span of the eigenvectors ϕ_k, $k \in \mathbb{N}^*$, and $T_S\varphi$, the tangent space to the sphere S at the point $\varphi \in S$. We say that system (5.3) is *spectral controllable in time T* if, for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$, there exists $v \in L^2((0,T),\mathbb{R}^n)$ such that the trajectory $\Psi(\cdot)$ of (5.3) starting at Ψ_0 satisfies $\Psi(T) = \Psi_f$. For system (5.4), that definition must be adapted as follows. Let $\langle .,. \rangle_{L^2}$ denote the $L^2(\Omega,\mathbb{C})$-scalar product. Then, system (5.4) is *spectral controllable in time T* if, for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$ with $\Im\langle\Psi_f, \psi_1(T)\rangle = \Im\langle\Psi_0, \psi_1(0)\rangle$ and for every $d_0 \in \mathbb{R}^n$, there exists $v \in L^2((0,T),\mathbb{R}^n)$ such that the trajectory $(\Psi, s, d)(\cdot)$ of (5.4) starting at $(\Psi_0, 0, d_0)$ satisfies $(\Psi, s, d)(T) = (\Psi_f, 0, 0)$.

Our main results deal with the spectral controllability of (5.3) and (5.4). Before describing them, let us make a general remark. Since we are dealing with controls only depending on time, the control systems under consideration can be put into the general form $\dot{x} = Ax + B(x)u$ where the state belongs to some $\mathbb{C}$-valued functional space X, the control u is $\mathbb{R}^n$-valued, the drift A is an (unbounded) linear operator admitting a complete orthonormal system of

eigenfunctions and the controlled vector field $B(\cdot)$ has rank one. Using the classical moment theory, it is easy to characterize two necessary conditions for spectral controllability in some finite time $T > 0$.

The first one corresponds to the Kalman condition for controllability in finite dimension. In our context, it means that

(Kal) for every eigenvalue λ of A, the projections $b_{k_j} := \langle \mu(q)\phi_1, \phi_{k_j} \rangle$, $1 \leq j \leq m(\lambda)$, of the controlled vector field $B(\cdot)$ on each eigenvector associated to λ are linearly independent in $\mathbb{R}^n$.

The above condition implies that the multiplicity of every eigenvalue λ of A is less than or equal to n. Note also that if A has simple spectrum (this will be referred as condition $(Simp)$), then condition (Kal) simply reads : the projections $b_k := \langle \mu(q)\phi_1, \phi_k \rangle$ of the controlled vector field $B(\cdot)$ on each (normalized) eigenvector is non zero. We refer to the latter condition as $(NonZ)$.

The second condition is specific to the infinite dimension (for the state space) and it is related to the minimality of the family $(e^{\pm i(\lambda_k - \lambda_1)t})_{k \in \mathbb{N}}$ in $L^2((0,T),\mathbb{C})$ (see Definition 5.9 below).

By applying a result of Haraux and Jaffard ([43]), we show that minimality never occurs in 3D for system (5.4) and also for system (5.3) if, in addition, the dipolar moment has a constant direction. In 2D, we show that minimality holds for both systems (5.3) and (5.4) if T is larger than a minimal time $T_{min}(\Omega)$. In turn, if the dipolar moment has a constant direction, spectral controllability in time $T > 0$ for system (5.4) enables one to define a Hilbert subspace H of $L^2(\Omega,\mathbb{C})$ in which (5.4) is controllable, with $L^2((0,T),\mathbb{R})$-controls, when $T > T_{min}(\Omega)$.

In order to get spectral controllability in time $T > T_{min}(\Omega)$, it therefore amounts, for a 2D domain Ω and a dipolar moment function μ, to check the validity of (Kal). Since the latter is difficult to verify for a given 2D domain Ω, we rather investigate conditions on the dipolar moment μ to insure that (Kal) holds true generically with respect to domains Ω with C^3 boundary. There is a trivial necessary condition on μ for (Kal) to hold true generically with respect to the domain : μ must be nowhere locally constant (NLC) i.e. its level sets are all of empty interior. (Indeed, simply consider a 2D domain where μ is constant. Then (Kal) does not hold, because of the $L^2(\Omega,\mathbb{C})$-orthogonality of the eigenvectors ϕ_k.) One of our main results says that condition (NLC) for a C^1 dipolar moment μ is also sufficient to prove that condition (Kal) holds true, generically with respect to domains Ω with C^3 boundary. To do so, we start from the well-known fact that the spectrum of the Laplacian operator on a domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary conditions is generically simple. Therefore, it amounts to prove that condition (NLC) for a C^1 dipolar moment μ is also sufficient for condition $(NonZ)$ to hold true, generically with respect to domains Ω with C^3 boundary. In summary, we can finally show that, in 2D, spectral controllability in finite time, for both systems (5.4) and (5.3) holds true, generically with respect to domains with C^3 boundary, if and only if the C^1 dipolar moment μ is nowhere locally constant.

Before giving the plan of the paper, let us sketch the argument showing that (NLC) implies $(NonZ)$, generically with respect to the domain. First of all, we must consider a topology for domains with C^3 boundary. Following [89], the latter is defined by taking as base of neighborhoods the sets $V(\Omega,\varepsilon)$ defined, for Ω any domain with C^3 boundary and $\varepsilon > 0$ small enough, as the images of Ω by $Id_2 + u$, $u \in W^{4,\infty}(\Omega,\mathbb{R}^2)$ and $\|u\|_{W^{4,\infty}} < \varepsilon$. We use $\mathbb{D}_3$ to denote the Banach space of domains with C^3 boundary equipped with the topology defined previously. A property is said to be generic in $\mathbb{D}_3$ if the subset of domains in $\mathbb{D}_3$ verifying that property is everywhere dense in $\mathbb{D}_3$.

We now fix a domain Ω with C^3 boundary and a C^1 dipolar moment μ verifying (NLC). Without loss of generality we assume that $(Simp)$ is verified by Ω and we first reduce the argument to showing, for every positive integer $k \geq 2$, the existence of a sequence (Ω_n) of domains with C^3 boundary converging to Ω such that $(NonZ)_k$ (i.e., $b_k \neq 0$ along the sequence (Ω_n)) holds true along the sequence. We proceed with a contradiction argument and we thus assume that there exists $\varepsilon > 0$ such that, for every $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$, the corresponding b_k is equal to zero. We compute the shape derivative of the relation $b_k = 0$ at $u \equiv 0$ and we can express it as an integral along the boundary of Ω, i.e.,

$$\int_{\partial\Omega} \langle u(q), \nu(q) \rangle M(q) d\sigma(q) = 0,$$

where ν denotes the outer unit normal vector field and $M(\cdot)$ is a $\mathbb{R}^2$-valued function defined on $\partial\Omega$. As we will see below, in order to define M, one must introduce ξ_1 and ξ_k, solutions of inhomogeneous Helmholtz equations (see (5.31) below). We at once deduce that $M(\cdot) \equiv 0$ on $\partial\Omega$. Reaching a contradiction in our argument amounts to show that the functions ξ_1, ξ_k introduced above actually do not exist. Unfortunately, we are not able to do that. By pushing further the contradiction argument, we compute the shape derivative of $b_k = 0$ at every $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$. That translates into the following relation : for $\varepsilon > 0$ small enough and for every $u, v \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$ and $\|v\|_{W^{4,\infty}} < \varepsilon$, one has

$$\int_{\partial(Id_2+u)\Omega} \langle v(q), \nu(u)(q) \rangle M(u)(q) d\sigma(q) = 0,$$

where $\nu(u)$ denotes the outer unit normal vector field defined on $\partial(Id_2 + u)(\Omega)$ and $M(u)(\cdot)$ is an $\mathbb{R}^2$-valued function defined on $\partial(Id_2 + u)(\Omega)$. The expression of $M(u)(\cdot)$ requires to define $\xi_1(u), \xi_k(u)$, solutions of inhomogeneous Helmholtz equations. Of course, $M(0), \xi_1(0)$ and $\xi_k(0)$ are equal to M, ξ_1 and ξ_k defined previously and we have that $M(u)(\cdot) \equiv 0$ on $\partial(Id_2 + u)(\Omega)$ for $\|u\|_{W^{4,\infty}} < \varepsilon$.

At this stage, we are again not able to derive a contradiction. So we again take the shape derivative of $M(u)(\cdot) \equiv 0$ on $\partial\Omega$ and end up with the relation

$$M'(u)(q) = -(u.\nu)(q) \frac{\partial M(0)}{\partial \nu}(q), \quad q \in \partial\Omega, \tag{5.5}$$

for $\|u\|_{W^{4,\infty}} < \varepsilon$. We now start a strategy first introduced in [26], which consists in defining $M'(u)$ for functions u defined on $\partial\Omega$ which are continuous except at some point $\bar{q}$ of $\partial\Omega$. For instance, we will take $u = u_{\bar{q}}$ as a Heaviside function $\mathcal{H}_0(\bar{q})$ admitting a single jump of discontinuity at an arbitrary point $q \in \partial\Omega$. The key remark is the following : if $(u.\nu)$ belongs to the Sobolev space $H^s(\partial\Omega)$ for some $s > 0$ then, by standard elliptic theory arguments, $M'(u)$ belongs to $H^{s-1}(\partial\Omega)$. In order to take advantage of the gap of regularity between the two sides of equation (5.5), we embark in the computation of the singular part of $M'(u_{\bar{q}})(\cdot)$ at $\bar{q}$ (in the distributional sense) and eventually come up with the following expression,

$$M'(u_{\bar{q}})(\sigma) = M_0 \ p.v. \left(\frac{1}{\sigma}\right) + R(\sigma),$$

where σ denotes the arclength (with $\sigma = 0$ corresponding to $\bar{q}$) and $R(\cdot)$ belongs to $H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon > 0$. Plugging back the above expression into equation (5.5), one must necessarily have $M_0 = 0$. Recalling that $\bar{q} \in \partial\Omega$ is arbitrary, we end up with $M_0(\cdot) \equiv 0$ on $\partial\Omega$. In [26], the previous relation on M_0 was providing additional information with respect to the relation

$M(u) \equiv 0$ on $\partial\Omega$, which allowed to conclude the contradiction argument. However, in the present situation, it turns out that $M_0(\cdot)$ is proportional to $M(0)(\cdot)$ and hence is trivially equal to zero. One must therefore compute the first non trivial term in the "singular" expansion of $M'(u_{\bar{q}}) + (u_{\bar{q}}.\nu)\frac{\partial M(0)}{\partial \nu}$ at $\bar{q}$, in the distributional sense. That procedure requires a detailed study of Dirichlet-to-Neumann operators associated to several Helmholtz equations. Once the non trivial term is characterized, we easily conclude.

The rest of the paper is organized as follows. In Section 5.2, we provide the main notations and precise definitions of the control systems (5.3) and (5.4), complete 1D results with their proofs and the statements of the main theorems of this article. Then, in Section 5.3, we give the proofs for the spectral controllability results in 2D and 3D. As for Section 5.4, the construction of some abstract spaces where we have 2D exact controllability is described. Section 5.5 contains the proof of the sufficiency of condition (NLC) to get generic controllability in 2D for the quantum box and Section 5.6 presents some conjectures. Finally, we gather in Appendix A the main results on shape differentiation used in the paper and Appendix B contains material on the Dirichlet-to-Neumann map for the Helmholtz equation with the proof of several technical lemmas which are needed in Section 5.5.

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5.2 Definitions, notations and statement of the results

5.2.1 Definition of the control problem

Let Ω be a domain of $\mathbb{R}^n$ (i.e. a bounded non empty open subset of $\mathbb{R}^n$), $n \in \{1, 2, 3\}$, with a C^1 boundary. We use $-\Delta_\Omega^D$ to denote the Laplacian operator on Ω with Dirichlet boundary conditions, i.e.

$$D(\Delta_\Omega^D) = H^2 \cap H_0^1(\Omega, \mathbb{C}), -\Delta_\Omega^D \phi = -\Delta\phi.$$

The space $L^2(\Omega, \mathbb{C})$ has a complete orthonormal system $(\phi_k)_{k\in\mathbb{N}^*}$ of eigenfunctions for $-\Delta_\Omega^D$,

$$\phi_k \in H^2 \cap H_0^1(\Omega, \mathbb{C}), -\Delta_\Omega^D \phi_k = \lambda_k \phi_k,$$

where $(\lambda_k)_{k\in\mathbb{N}^*}$ is a non-decreasing sequence of positive real numbers. With this notation, the eigenvalues λ_k are counted as many times as their multiplicity. For $t \in \mathbb{R}$ and $q \in \Omega$, we define the function ψ_1 by

$$\psi_1(t, q) := \phi_1(q) e^{-i\lambda_1 t}.$$

We recall that $-i\Delta_\Omega^D$ generates a C^0-group of isometries of $L^2(\Omega, \mathbb{C})$ defined by

$$e^{-i\Delta t}\varphi := \sum_{k\in\mathbb{N}^*} \langle \varphi, \phi_k \rangle e^{-i\lambda_k t} \phi_k, \forall \varphi \in L^2(\Omega, \mathbb{C}).$$

In this paper, we study controllability properties of the linear systems (5.3) and (5.4).

In order to consider them as control systems, we first need a concept of trajectories associated to these systems. For that purpose, recall that the unit sphere S of $L^2(\Omega, \mathbb{C})$ is defined as follows,

$$S := \{\varphi \in L^2(\Omega, \mathbb{C}); \|\varphi\|_{L^2(\Omega)} = 1\},$$

and, for $\varphi \in S$, the tangent space to the sphere S at the point φ is given by

$$T_S\varphi := \left\{ \xi \in L^2(\Omega, \mathbb{C}); \Re\left(\int_\Omega \xi(q)\overline{\varphi(q)} dq \right) = 0 \right\}.$$

Définition 5.1. (Weak solutions) Let $T > 0$, $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$, $\Psi_0 \in T_S\phi_1$ and $v \in L^1((0,T),\mathbb{R}^n)$. A weak solution to the Cauchy problem

$$\begin{cases} i\frac{\partial \Psi}{\partial t}(t,q) = -\Delta\Psi(t,q) - \langle v(t), \mu(q)\rangle \psi_1(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \\ \Psi(0) = \Psi_0, \end{cases} \tag{5.6}$$

is a function $\Psi \in C^0([0,T], L^2(\Omega,\mathbb{C}))$ such that for every $t \in [0,T]$,

$$\Psi(t) = e^{i\Delta t}\Psi_0 + i\int_0^t e^{i\Delta(t-s)}[\langle v(s), \mu\rangle \psi_1(s)]ds \text{ in } L^2(\Omega,\mathbb{C}). \tag{5.7}$$

Then (Ψ, v) is a trajectory of the control system (5.3) on $[0,T]$.

Let $s_0, d_0 \in \mathbb{R}^n$. A weak solution to the Cauchy problem

$$\begin{cases} i\frac{\partial \Psi}{\partial t}(t,q) = -\Delta\Psi(t,q) - \langle v(t), \mu(q)\rangle \psi_1(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \\ \Psi(0) = \Psi_0, \\ \dot{s}(t) = v(t), \ s(0) = s_0, \\ \dot{d}(t) = s(t), \ d(0) = d_0, \end{cases} \tag{5.8}$$

is a function (Ψ, s, d) with $s \in W^{1,1}((0,T),\mathbb{R}^n)$, $d \in W^{2,1}((0,T),\mathbb{R}^n)$ solutions of

$$\begin{aligned} \dot{s}(t) &= v(t) \ \text{ in } L^1((0,T),\mathbb{R}^n), \ s(0) = s_0, \\ \dot{d}(t) &= s(t) \ \text{ in } L^1((0,T),\mathbb{R}^n), \ d(0) = d_0, \end{aligned}$$

and $\Psi \in C^0([0,T], L^2(\Omega,\mathbb{C}))$ such that for every $t \in [0,T]$, (5.7) holds. Then $((\Psi, s, d), v)$ is a trajectory of the control system (5.4) on $[0,T]$.

The following proposition recalls a classical existence and uniqueness result for the solutions of (5.6), from which one can deduce the similar result for (5.8).

Theorem 5.1. *For every $T > 0$, $\Psi_0 \in T_S\phi_1$, $v \in L^1((0,T),\mathbb{R}^n)$, there exists a unique weak solution to the Cauchy problem (5.6) and $\Psi(t) \in T_S\psi_1(t)$ for every $t \geq 0$.*

Then, the system (5.3) is a control system where
– the state is the function Ψ, with $\Psi(t) \in T_S\psi_1(t)$ for every $t \in \mathbb{R}_+$,
– the control is $v : t \in \mathbb{R}_+ \mapsto v(t) \in \mathbb{R}^n$, $L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ is the set of admissible controls

and the system (5.4) is a control system where
– the state is the triple (Ψ, s, d), with $\Psi(t) \in T_S\psi_1(t)$ for every $t \in \mathbb{R}_+$,
– the control is $v : t \in \mathbb{R}_+ \mapsto v(t) \in \mathbb{R}^n$ and $L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ is the set of admissible controls.

More precisely, in this paper, we investigate the following controllability property for (5.3).

Définition 5.2 (Spectral controllability for (5.3))**.** The system (5.3) is spectral controllable in time T if, for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$, there exists $v \in L^2((0,T),\mathbb{R}^n)$ such that the solution of (5.6) satisfies $\Psi(T) = \Psi_f$, where

$$\mathcal{D} := \text{Span}\{\phi_k; k \in \mathbb{N}^*\}.$$

For the system (5.4), this definition needs to be adapted because of the presence of s and d in the state variable and because the directions $\Im\langle\Psi(t), \psi_1(t)\rangle$ and $s(t)$ are linked. Indeed, any solution of (5.8) satisfies

$$\Im\langle\Psi(t), \psi_1(t)\rangle = \Im\langle\Psi_0, \psi_1(0)\rangle + \sum_{j=1}^{n} \langle\mu^{(j)}\phi_1, \phi_1\rangle[s^{(j)}(t) - s^{(j)}(0)], \forall t, \tag{5.9}$$

where, for $x \in \mathbb{R}^n$, $x^{(j)}$ denotes its components, $x = (x^{(1)}, ..., x^{(n)})$ and $\langle ., . \rangle$ denotes the $L^2(\Omega, \mathbb{C})$-scalar product. Therefore, we study the following controllability property for (5.4).

Définition 5.3 (Spectral controllability for (5.4))**.** The system (5.4) is spectral controllable in time T if for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$ with $\Im\langle \Psi_f, \psi_1(T)\rangle = \Im\langle \Psi_0, \psi_1(0)\rangle$, for every $d_0 \in \mathbb{R}^n$, there exists $v \in L^2((0,T), \mathbb{R}^n)$ such that the solution of (5.8) with $s_0 = 0$ satisfies $(\Psi, s, d)(T) = (\Psi_f, 0, 0)$.

The notations Ω, $n \in \{1, 2, 3\}$, ϕ_k, ψ_1, $\langle ., . \rangle$, S, T_S, $\mathcal{D}$, $x = (x^{(1)}, ..., x^{(n)}) \in \mathbb{R}^n$ introduced in this section are used all along this article. We also denote $(e_j)_{1 \leq j \leq n}$ the canonical basis of $\mathbb{R}^n$ and $\omega_k := \lambda_k - \lambda_1$, for every $k \in \mathbb{N}^*$. We use the same notation for the $\mathbb{R}^n$-scalar product and the $L^2(\Omega)$-scalar product but if a confusion is possible we precise the space in subscript $\langle ., . \rangle_{L^2(\Omega)}$ or $\langle ., . \rangle_{\mathbb{R}^n}$. When some confusion is possible, we also precise the domain on the eigenvalues and eigenfunctions of the Laplacian : λ_k^Ω, ϕ_k^Ω.

5.2.2 Previous 1D results, difficulties of the 2D and 3D generalizations

In this section, we recall classical results about the controllability of the systems (5.3) and (5.4) in 1D, that are the starting point of the strategies developed in [9] and [10] for the nonlinear systems (5.1) and (5.2). We also give their proof in order to explain the difficulties arising in their generalization to the 2D and 3D cases.

We take $\Omega = (0, 1)$, so

$$\phi_k(q) = \sqrt{2}\sin(k\pi q), \lambda_k = (k\pi)^2$$

and we use the following notations

$$H^s_{(0)}((0,1), \mathbb{C}) := D(A^{s/2}) \text{ where } D(A) := H^2 \cap H^1_0((0,1), \mathbb{C}), A\varphi := -\varphi''.$$

5.2.2.1 1D controllability of (5.3)

For the control system (5.3), we have the following result.

Proposition 5.2. *Let $\Omega = (0, 1)$ and $\mu \in W^{3,\infty}((0,1), \mathbb{R})$.*

(1) *We assume that*

$$\exists c_1, c_2 > 0, \frac{c_1}{k^3} \leq |\langle \mu\phi_1, \phi_k\rangle| \leq \frac{c_2}{k^3}, \forall k \in \mathbb{N}^*. \tag{5.10}$$

Then, for every $T > 0$, the system (5.3) is controllable in $H^3_{(0)}((0,1), \mathbb{C})$ with control functions in $L^2((0,T), \mathbb{R})$: for every $T > 0$, $\Psi_0, \Psi_f \in H^3_{(0)}((0,1), \mathbb{C})$ with $\Psi_0 \in T_S\psi_1(0)$ and $\Psi_f \in T_S\psi_1(T)$, there exists $v \in L^2((0,T), \mathbb{R})$ such that the solution of (5.6) satisfies $\Psi(T) = \Psi_f$.

(2) *We assume that there exists $m \in \mathbb{N}^*$ such that $\langle \mu\phi_1, \phi_m\rangle = 0$ and*

$$\exists c_1, c_2 > 0, \frac{c_1}{k^3} \leq |\langle \mu\phi_1, \phi_k\rangle| \leq \frac{c_2}{k^3}, \forall k \in \mathbb{N}^* \text{ such that } \langle \mu\phi_1, \phi_k\rangle \neq 0. \tag{5.11}$$

Then, the system (5.3) is not controllable : for every $T > 0$, $\Psi_0 \in L^2((0,1), \mathbb{C})$ and $v \in L^1((0,T), \mathbb{R})$ the solution of (5.6) satisfies

$$\langle \Psi(T), \phi_k\rangle = \langle \Psi_0, \phi_k\rangle e^{-i\lambda_k T}, \forall k \in \mathbb{N}^* \text{ such that } \langle \mu\phi_1, \phi_k\rangle = 0.$$

But one can characterize the reachable set : for every $T > 0$, $\Psi_0, \Psi_f \in H^3_{(0)}((0,1), \mathbb{C})$ with $\Psi_0 \in T_S\psi_1(0)$, $\Psi_f \in T_S\psi_1(T)$, $\langle \Psi_f, \phi_k\rangle = \langle \Psi_0, \phi_k\rangle e^{-i\lambda_k T}$ for every $k \in \mathbb{N}^*$ such that $\langle \mu\phi_1, \phi_k\rangle = 0$, there exists $v \in L^2((0,T), \mathbb{R})$ such that the solution of (5.6) satisfies $\Psi(T) = \Psi_f$.

Remark 5.1. Let us emphasize that the assumption (5.10) is generic with respect to $\mu \in W^{3,\infty}((0,1),\mathbb{R})$. Indeed, thanks to Baire's Lemma, it is easy to prove that the property "$\langle \mu\phi_1, \phi_k\rangle \neq 0$, $\forall k \in \mathbb{N}^*$"holds generically with respect to $\mu \in W^{3,\infty}((0,1),\mathbb{R})$. Moreover, for such a function μ, integrations by parts lead to

$$\langle \mu\phi_1, \phi_k\rangle = 2\int_0^1 \mu(q)\sin(\pi q)\sin(k\pi q)dq = \frac{4k[(-1)^{k+1}\mu'(1) - \mu'(0)]}{(k^2-1)^2} + o\left(\frac{1}{k^3}\right).$$

Thus, the asymptotic behavior in $1/k^3$ of these coefficients is equivalent to the property "$\mu'(1) \pm \mu'(0) \neq 0$", that is also generic in $W^{3,\infty}((0,1),\mathbb{R})$.

The key ingredient for the proof of Proposition 5.2 is the following theorem due to Kahane [56, Theorem III.6.1, p. 114].

Theorem 5.3. *Let $(\mu_k)_{k\in\mathbb{N}^*} \subset \mathbb{R}$ such that $\mu_1 = 0$ and*

$$\mu_{k+1} - \mu_k \geq \rho > 0, \forall k \in \mathbb{N}^*. \tag{5.12}$$

Let $T > 0$ be such that

$$\lim_{x\to+\infty} \frac{N(x)}{x} < \frac{T}{2\pi},$$

where, for $x > 0$, $N(x)$ is the largest number of μ_k's contained in an interval of length x. Then, there exists $C > 0$ such that, for every $c = (c_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*,\mathbb{C})$ with $c_1 \in \mathbb{R}$, there exists $w \in L^2((0,T),\mathbb{R})$ such that $\|w\|_{L^2((0,T),\mathbb{R})} \leq C\|c\|_{l^2(\mathbb{N}^*,\mathbb{C})}$ and

$$\int_0^T w(t)e^{i\mu_k t}dt = c_k, \forall k \in \mathbb{N}^*.$$

Remark 5.2. The proof of Theorem 5.3 relies on an Ingham inequality for the family

$$\{1, e^{i\mu_k t}, e^{-i\mu_k t}; k \in \mathbb{N}^*, k \geq 2\},$$

which corresponds to the Riesz basis property of this family in $L^2((0,T),\mathbb{C})$. For the proof of Theorem 5.3, see, for example Krabs [61, Section 1.2.2], Komornik and Loreti [60, Chapter 9], or Avdonin and Ivanov [6, Chapter II Section 4]. For the proof of similar results, we also refer to the prior works by Ingham [48], and to Beurling [14, p. 341-365], Haraux [42], Redheffer [83], Russel [86, Section 3], Schwartz [87]. Improvements of Theorem 5.3 have been obtained by Jaffard, Tucsnak and Zuazua [50], [51], Jaffard and Micu [49], Baiocchi, Komornik and Loreti [7], Komornik and Loreti [59], [60, Theorem 9.4, p. 177].

Proof of Proposition 5.2. We assume (5.10). Let $T > 0$ and $\Psi_0 \in T_S\psi_1(0)$. By definition, the weak solution of (5.6) with some control $v \in L^2((0,T),\mathbb{R})$ is

$$\Psi(t,q) = \sum_{k=1}^{\infty} x_k(t)\phi_k(q),$$

where

$$x_k(t) = \Big(\langle \Psi_0, \phi_k\rangle + i\langle \mu\phi_1, \phi_k\rangle \int_0^t v(\tau)e^{i\omega_k\tau}d\tau\Big)e^{-i\lambda_k t}, \ \forall\ k \in \mathbb{N}^*,$$

with convergence in $L^2((0,1),\mathbb{C})$ for every $t \in [0,T]$, where $\omega_k := \lambda_k - \lambda_1$, for every $k \in \mathbb{N}^*$. Since $\langle \mu\phi_1, \phi_k\rangle \neq 0$, for every $k \in \mathbb{N}^*$, the equality $\Psi(T) = \Psi_f$ in $L^2((0,1),\mathbb{C})$ is equivalent to the following trigonometric moment problem on the control v,

$$\int_0^T v(t)e^{i\omega_k t}dt = d_k, \forall k \in \mathbb{N}^*, \tag{5.13}$$

where

$$d_k := \frac{\langle \Psi_f, \phi_k\rangle e^{i\lambda_k T} - \langle \Psi_0, \phi_k\rangle}{i\langle \mu\phi_1, \phi_k\rangle}, \forall k \in \mathbb{N}^*. \tag{5.14}$$

Thanks to (5.10), the right-hand side $(d_k)_{k\in\mathbb{N}^*}$ belongs to $l^2(\mathbb{N}^*, \mathbb{C})$ if and only if $\Psi_f - e^{-iAT}\Psi_0 \in H^3_{(0)}((0,1),\mathbb{C})$, and in that case, (5.13) has a solution $v \in L^2((0,T),\mathbb{R})$ for every $T > 0$, thanks to Theorem 5.3. The proof of the statement **(2)** is similar.

□

Now, let us discuss the generalization of Proposition 5.2 to the 2D and 3D cases. In 2D and 3D, the equality $\Psi(T) = \Psi_f$ for a solution of (5.6) is equivalent to

$$i\Big\langle \langle \mu\phi_1, \phi_k\rangle_{L^2(\Omega)}, \int_0^T v(t)e^{i\omega_k t}dt\Big\rangle_{\mathbb{R}^n} = \langle \Psi_f, \phi_k\rangle e^{i\lambda_k T} - \langle \Psi_0, \phi_k\rangle, \forall k \in \mathbb{N}^*. \tag{5.15}$$

Thus, the property

$$\langle \mu\phi_1, \phi_k\rangle \neq 0, \forall k \in \mathbb{N}^*$$

is still a necessary condition for the controllability of (5.3). Let us assume that this property holds, then (5.15) is satisfied in particular when

$$\int_0^T v(t)e^{i\omega_k t}dt = -i\frac{\langle \mu\phi_1, \phi_k\rangle}{|\langle \mu\phi_1, \phi_k\rangle|^2}\Big(\langle \Psi_f, \phi_k\rangle e^{i\lambda_k T} - \langle \Psi_0, \phi_k\rangle\Big), \forall k \in \mathbb{N}^*.$$

Thus, the controllability of (5.3) can be reduced to the solvability of n trigonometric moment problems on the real valued functions $v^{(1)},...,v^{(n)}$.

In 2D, the existence of a regular domain Ω of $\mathbb{R}^2$ such that the eigenvalues of Δ^D_Ω present a uniform gap (which corresponds to the assumption (5.12)) is an open problem. For general 2D regular domains, we only have Weyl's Formula, $\exists c = c(\Omega) > 0$, $\exists \alpha = \alpha(\Omega) \in (0,1)$ such that

$$\sharp\{k \in \mathbb{N}^*; \lambda_k \in [0,t]\} = ct + O(t^\alpha), \text{ when } t \to +\infty.$$

This formula is not sufficient to ensure the existence of a uniform gap between the frequencies ω_k. Therefore the classical result given in Theorem 5.3 cannot be applied : the controllability of (5.3) is a more difficult problem in 2D than in 1D.

In 3D, with Weyl's formula, $\exists c = c(\Omega) > 0, \exists \alpha = \alpha(\Omega) \in (0, 3/2)$ such that

$$\sharp\{k \in \mathbb{N}^*; \lambda_k \in [0,t]\} = ct^{3/2} + O(t^\alpha), \quad \text{when } t \to +\infty,$$

no uniform gap is possible. Thus, the non controllability of (5.3) is expected.

The exact controllability of (5.3) in 2D and 3D being a difficult problem, it is natural to study a weaker controllability property for this system. This is why we investigate its spectral controllability in this article. Notice that the spectral controllability in time T of (5.3) is equivalent to the existence of a solution $v \in L^2((0,T),\mathbb{R}^n)$ of (5.15) for any right hand side with finite support. This remark will be used in the study of the spectral controllability of (5.3) (see Section 5.3.2).

5.2.2.2 1D controllability of (5.4)

For the control system (5.4), we have the following result.

Proposition 5.4. *Let $\Omega = (0,1)$ and $\mu \in W^{3,\infty}((0,1),\mathbb{R})$.*

***(1)** The system (5.4) is not controllable : for every $\Psi_0 \in T_S\psi_1(0)$, $s_0, d_0 \in \mathbb{R}$, $v \in L^1_{loc}(\mathbb{R}_+,\mathbb{R})$, the solution of (5.8) satisfies (5.9).*

***(2)** If (5.11) holds, then, one can characterize the reachable set for (5.4) : for every $T>0$, $\Psi_0, \Psi_f \in H^3_{(0)}((0,1),\mathbb{C})$, $s_0, s_f, d_0, d_f \in \mathbb{R}$ with $\langle \Psi_f, \psi_1(T)\rangle = \langle \Psi_0, \psi_1(0)\rangle + i\langle \mu\phi_1, \phi_1\rangle(s_f - s_0)$ and*

$$\langle \Psi(T), \phi_k\rangle = \langle \Psi_0, \phi_k\rangle e^{-i\lambda_k T}, \forall k \geq 2 \text{ such that } \langle \mu\phi_1, \phi_k\rangle = 0, \tag{5.16}$$

there exists $v \in L^2((0,T),\mathbb{R})$ such that the solution of (5.8) satisfies $(\Psi, s, d)(T) = (\Psi_f, s_f, d_f)$.

The proof of this proposition is similar to the one of Proposition 5.2.

Notice that, in 2D and 3D, the equality $(\Psi, s, d)(T) = (\Psi_f, 0, 0)$ for the solution of (5.8) with $s_0 = 0$, $\Psi_0 \in T_S\psi_1(0)$, $\Psi_f \in T_S\psi_1(T)$ such that $\Im\langle \Psi_0, \psi_1(0)\rangle = \Im\langle \Psi_f, \psi_1(T)\rangle$ is equivalent to

$$\begin{cases} i\left\langle \langle \mu\phi_1, \phi_k\rangle_{L^2(\Omega)}, \int_0^T v(t)e^{i\omega_k t}dt \right\rangle_{\mathbb{R}^n} = \langle \Psi_f, \phi_k\rangle e^{i\lambda_k T} - \langle \Psi_0, \phi_k\rangle, \forall k \geq 2, \\ \int_0^T v(t)dt = 0, \\ \int_0^T tv(t)dt = d_0. \end{cases} \tag{5.17}$$

Thus, the spectral controllability in time T of (5.4) is equivalent to the existence of a solution $v \in L^2((0,T),\mathbb{R}^n)$ of (5.17), for any right hand side with finite support. This remark will be used in the study of the spectral controllability of (5.4) (see Section 5.3.3).

5.2.3 Statement of the main results

In order to state our results, we first give several definitions relative to the domain and the dipolar moment.

Définition 5.4 (Kalman condition, (Kal))**.** Let Ω be a domain of $\mathbb{R}^n$, $n = 2,3$ with C^1 boundary. Then Ω verifies Property (Kal) if

(Kal) any eigenvalue λ of $-\Delta_\Omega^D$ has a multiplicity $m \leq n$ and the vectors $\langle \mu\phi_1, \phi_{k_1}\rangle, ..., \langle \mu\phi_1, \phi_{k_m}\rangle$ are linearly independant in $\mathbb{R}^n$, where $k_1 < ... < k_m$ and $\phi_{k_1}, ..., \phi_{k_m}$ are the eigenvectors associated to λ.

Définition 5.5 (Simplicity of the spectrum, $(Simp)$)**.** Let Ω be a domain of $\mathbb{R}^n$, $n = 2,3$ with C^1 boundary. Then Ω verifies Property $(Simp)$ if

$$(Simp) \quad \text{the eigenvalues of } -\Delta_\Omega^D \text{ are simple.}$$

Définition 5.6 (Non zero projection, $(NonZ)$)**.** Consider $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$, $n = 2,3$ and $(\phi_k)_{k\in\mathbb{N}^*}$ the complete orthonormal system of eigenvectors of $-\Delta_\Omega^D$. Then $\mu\phi_1$ has a non zero projection on $(\phi_k)_{k\in\mathbb{N}^*}$ if, for every integer $k \geq 2$, we have

$$(NonZ)_k \quad \langle \mu\phi_1, \phi_k\rangle \neq 0.$$

In that case, we say that μ verifies Property $(NonZ)$.

Remark that if a domain Ω satisfies $(Simp)$, then condition (Kal) reduces to condition $(NonZ)$. The next theorem gathers our result regarding the spectral controllability properties for system (5.3).

Theorem 5.5. *(1) Let Ω be a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ verifying (Kal). Then, there exists $T_{min} = T_{min}(\Omega) > 0$ such that*

(1.a) *for every $T > T_{min}$, system (5.3) is spectral controllable in time T ;*

(1.b) *for every $T < T_{min}$, system (5.3) is not spectral controllable in time T, under the additional assumption*

$$\mu(x) = \tilde{\mu}(x)e_1 \text{ where } \tilde{\mu} \in C^0(\overline{\Omega}, \mathbb{R}). \tag{5.18}$$

(2) Let Ω be a domain of $\mathbb{R}^n$, $n = 2, 3$, with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$ such that (Kal) is not verified. Then, system (5.3) is not spectral controllable.

(3) Let Ω be a domain of $\mathbb{R}^3$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^3)$ of the form (5.18). Then, system (5.3) is not spectral controllable.

Remark 5.3. Let us emphasize that (Kal) holds true generically with respect to the pair (Ω, μ) because conditions $(Simp)$ and $(NonZ)$ hold true simultaneously generically with respect to the pair (Ω, μ), where Ω is a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$. Indeed, the genericity of $(Simp)$ with respect to the domain Ω is a classical result (see for instance [45]). Moreover, for a domain Ω of $\mathbb{R}^2$ with C^1 boundary verifying $(Simp)$, the set

$$\{\mu \in C^0(\overline{\Omega}, \mathbb{R}^2); \langle \mu\phi_1, \phi_k \rangle \neq 0, \forall k \in \mathbb{N}^*\}$$

is dense in $C^0(\overline{\Omega}, \mathbb{R}^2)$ (it can be proved thanks to Baire's Lemma).

As for system (5.4), we prove the following result.

Theorem 5.6. *(1) Let Ω be a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ verifying (Kal). Let $T_{min} = T_{min}(\Omega)$ be as in Theorem 5.5. Then, system (5.4) is spectral controllable in time $T > T_{min}$.*

(2) Let Ω be a domain of $\mathbb{R}^n$, $n = 2, 3$, with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$ such that (Kal) is not verified. Then, system (5.4) is not spectral controllable.

(3) Let Ω be a domain of $\mathbb{R}^3$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^3)$. Then system (5.4) is not spectral controllable : for every $T > 0$ and $m \in \mathbb{N}^*$, there exists $d_0 \in \mathbb{R}^3$ such that $(i\phi_m, 0, d_0)$ is not zero controllable in time T.

Remark 5.4. Notice that in Item **(3)** of Theorem 5.6, the dipolar moment μ is not necessarily one dimensional. Thus, we prove a stronger non controllability result for this 3D system, than the one given in Theorem 5.5 **(3)**. This improvement is due to the presence of s and d in the state variable.

The proofs of Theorems 5.5 and 5.6 are given in Section 5.3.

In Section 5.4, we prove that, one can recover the exact controllability, in some abstract spaces, for the system (5.3) in 2D with μ of the form (5.18) thanks to the previous spectral controllability result. Such abstract spaces may be used for the study of the nonlinear system. This is an open problem.

According to Theorem 5.6, one knows that, in 2D, property (Kal) is a necessary and sufficient condition for the spectral controllability of (5.3) and (5.4) in time $T > T_{min}(\Omega)$.

We next use that characterization to prove that spectral controllability of (5.3) and (5.4) in large time holds true generically with respect to the 2D domain Ω. For that purpose, let us first precise the topology on domains we are using, then define genericity and finally state the condition on the dipolar moment μ that ensures the genericity.

For $l \geq 1$, the set $\mathbb{D}_l$ of domains Ω of $\mathbb{R}^2$ with C^l boundary. Following [89], we define next a topology on $\mathbb{D}_l$. Consider the Banach space $W^{l+1,\infty}(\Omega, \mathbb{R}^2)$ equipped with its standard norm. For $\Omega \in \mathbb{D}_l$, $u \in W^{l+1,\infty}(\Omega, \mathbb{R}^2)$, let $\Omega + u := (\mathrm{Id} + u)(\Omega)$ be the subset of points $y \in \mathbb{R}^2$ such that $y = x + u(x)$ for some $x \in \Omega$ and $\partial\Omega + u := (\mathrm{Id} + u)(\partial\Omega)$ its boundary. For $\varepsilon > 0$, let $V(\Omega, \epsilon)$ be the set of all $\Omega + u$ with $u \in W^{l+1,\infty}(\Omega, \mathbb{R}^2)$ and $\|u\|_{W^{l+1,\infty}} \leq \varepsilon$. The topology of $\mathbb{D}_l$ is defined by taking the sets $V(\Omega, \varepsilon)$ with ε small enough as a base of neighborhoods of Ω. Then, $\mathbb{D}_l$ is a Banach space.

Définition 5.7. We say that a property (P) is generic in $\mathbb{D}_l$ if the set of domains of $\mathbb{D}_l$ on which this property holds true is dense in $\mathbb{D}_l$: for every $\Omega \in \mathbb{D}_l$, there exists $\rho > 0$ such that the set $\{u \in E_\rho(\Omega); \Omega + u \text{ satisfies } (P)\}$ is dense in $E_\rho(\Omega)$, where $E_\rho(\Omega) := \{u \in W^{l+1,\infty}(\Omega, \mathbb{R}^2); \|u\|_{W^{l+1,\infty}} < \rho\}$.

Définition 5.8 (Non locally constant, (NLC))**.** A map $\mu \in C^0(\mathbb{R}^2, \mathbb{R}^2)$ is said to be *nowhere locally constant* if, for every $\mu_0 \in \mathbb{R}^2$, the level set $\{q \in \mathbb{R}^2 \ \ \mu(q) = \mu_0\}$ has an empty interior.

Note that if μ is (NLC) and continuously differentiable, then the subset of $\mathbb{R}^n$, $n = 2, 3$, where the differential of μ is not zero, must be open and dense.

We now state one of the main results of the paper.

Theorem 5.7. *Let $\mu \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. The spectral controllability in large time for system (5.4) is generic in $\mathbb{D}_3$ if and only if μ is nowhere locally constant.*

According to Item **(2)** of Theorem 5.6, the proof of the previous theorem reduces to establishing the next proposition, since $(Simp)$ and $(NonZ)$ both verified imply that (Kal) holds true.

Proposition 5.8. *Let $\mu \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. If $\Omega \in \mathbb{D}_1$, we say that Ω has property (A) if $(Simp)$ and $(NonZ)$ hold true for Ω. Then, property (A) is generic in $\mathbb{D}_3$ if and only if μ is nowhere locally constant.*

Section 5.5 is devoted to the proof of the above proposition.

5.3 Spectral controllability in 2D and 3D

The goal of this Section is the proof of Theorems 5.5 and 5.6. This section is organized as follows.

In Subsection 5.3.1, we state a sufficient condition for the minimality in $L^2((0,T), \mathbb{C})$ of a family of complex exponentials. This condition, due to Haraux and Jaffard [43], involves Weyl's formula.

In Subsection 5.3.2, we prove Theorem 5.5, thanks to Haraux and Jaffard 's result.

In Subsection 5.3.3, we prove Theorem 5.6. The proofs of the two first statements also rely on Haraux and Jaffard's result. The proof of the third statement involves different ideas, about the set of zeros of holomorphic functions.

5.3.1 Haraux and Jaffard 's result

First, let us recall the definition of the minimality of a family of vectors.

Définition 5.9. Let X be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A family $(z_k)_{k\in\mathbb{Z}}$ of vectors of X is minimal in X if, for every $m \in \mathbb{Z}$, z_m does not belong to the closure in X of the vector space generated by $\{z_k; k \in \mathbb{Z} - \{m\}\}$,

$$z_m \notin \mathrm{Cl}_X\Big(\mathrm{Span}\{z_k; k \in \mathbb{Z} - \{m\}\}\Big), \forall m \in \mathbb{Z}.$$

When X is an Hilbert space, we have the following classical equivalent definitions.

Proposition 5.9. *Let $(X, \langle ., . \rangle_X)$ be a Hilbert space and $(z_k)_{k\in\mathbb{Z}}$ be a family of vectors of X. The following statements are equivalent.*

(1) *$(z_k)_{k\in\mathbb{Z}}$ is minimal in X.*

(2) *For every $m \in \mathbb{Z}$, there exists $C_m > 0$ such that, for every $f \in X$ of the form $f = \sum_{k\in K} f_k z_k$ where $K \subset \mathbb{Z}$ is finite,*

$$C_m |f_m| \leq \|f\|_X.$$

(3) *There exists a family $(Z_k)_{k\in\mathbb{Z}}$ of vectors of X bi-orthogonal to $(z_k)_{k\in\mathbb{Z}}$, i.e.*

$$\langle z_m, Z_k \rangle_X = \delta_{m,k}, \forall k, m \in \mathbb{Z}.$$

(4) *For every $(d_k)_{k\in\mathbb{Z}} \subset \mathbb{K}$ with finite support, there exists $v \in X$ solution of the moment problem*

$$\langle v, z_k \rangle_X = d_k, \forall k \in \mathbb{Z}. \tag{5.19}$$

Proof of Proposition 5.9. For **(1)** $\Rightarrow$ **(2)**, the largest value for the constant C_m is

$$C_m := \mathrm{dist}\Big(z_m, \mathrm{Span}\{z_k; k \in \mathbb{Z} - \{m\}\}\Big).$$

The implications **(2)** $\Rightarrow$ **(1)**, **no (1)** $\Rightarrow$ **no (3)** and **(3)** $\Leftrightarrow$ **(4)** are easy. For **(1)** $\Rightarrow$ **(3)**, one can take

$$Z_m := P_m\Big(\frac{z_m}{\|z_m\|_X^2}\Big)$$

where P_m is the orthogonal projection from X to $V_m^\perp$, the orthogonal supplementary of $V_m := \mathrm{Cl}_X(\mathrm{Span}\{z_k; k \in \mathbb{Z} - \{m\}\})$ in X, which is a closed vector subspace of X.

□

Remark 5.5. The statement **(4)** is particularly important in this article. Indeed, as seen in Section 5.2.2, the spectral controllability in time T of (5.3) is equivalent to the solvability of a moment problem of the form (5.19) with $X = L^2((0,T), \mathbb{R}^n)$, $z_0 := \langle \mu\phi_1, \phi_1 \rangle$, $z_k := \langle \mu\phi_1, \phi_{k+1} \rangle \cos(\omega_{k+1}t)$, $z_{-k} := \langle \mu\phi_1, \phi_{k+1} \rangle \sin(\omega_{k+1}t)$, $\forall k \in \mathbb{N}^*$. Thus, the spectral controllability in time T of (5.3) is equivalent to the minimality of the familly $(z_k)_{k\in\mathbb{Z}}$ in $L^2((0,T), \mathbb{R}^n)$.

The following theorem is the key point of Section 5.3. It has been proved by Haraux and Jaffard in [43, Corollary 2.3.5], as a consequence of the Beurling Malliavin Theorem, thanks to the computation of the Beurling-Malliavin density of a sequence that satisfies Weyl's formula.

Theorem 5.10. *Let $(\mu_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers such that*

$$\sharp\{k\in\mathbb{Z}; 0\le\mu_k\le t\}=dt+O(t^\alpha),\quad \sharp\{k\in\mathbb{Z}; -t\le\mu_k\le 0\}=dt+O(t^\alpha),$$

for some $d\ge 0$ and $\alpha\in(0,1)$. Then,
(1) for every $T>2\pi d$, the family $\{e^{i\mu_k t}; k\in\mathbb{Z}\}$ is minimal in $L^2((0,T),\mathbb{C})$,
(2) for every $T<2\pi d$, the family $\{e^{i\mu_k t}; k\in\mathbb{Z}\}$ is not minimal in $L^2((0,T),\mathbb{C})$.

Remark 5.6. Notice that, when $\mu_0=0$ and $\mu_k=-\mu_{-k}>0$, for every $k\in\mathbb{N}^*$, then the minimality of the family $\{e^{i\mu_k t}; k\in\mathbb{Z}\}$ in $L^2((0,T),\mathbb{C})$ is equivalent to the minimality of the family $\{1,\cos(\mu_k t),\sin(\mu_k t); k\ge 0\}$ in $L^2((0,T),\mathbb{R})$.

5.3.2 Proof of Theorem 5.5

The goal of this section is the proof of Theorem 5.5. thanks to Theorem 5.10.

Proof of Theorem 5.5. **(1)** Let Ω be a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu\in C^0(\overline{\Omega},\mathbb{R}^2)$ be such that (Kal) holds. Thanks to Weyl's formula, there exists $d=d(\Omega)\in(0,+\infty)$ and $\alpha=\alpha(\Omega)\in(0,1)$ such that

$$\sharp\{k\in\mathbb{N}^*;\omega_k\in[0,t]\}=dt+O(t^\alpha)\text{ when }t\to+\infty. \tag{5.20}$$

Let $T_{min}=T_{min}(\Omega):=2\pi d$.

(1.a) Let $T>T_{min}$, $\Psi_0\in\mathcal{D}\cap T_S\psi_1(0)$, $\Psi_f\in\mathcal{D}\cap T_S\psi_1(T)$ and let us prove that there exists $v\in L^2((0,T),\mathbb{R}^2)$ solution of (5.15). We introduce

$$\Lambda_1:=\{k\in\mathbb{N}^*;\lambda_k\text{ is a simple eigenvalue of }\Delta_\Omega^D\},\quad \Lambda_2:=\{k\in\mathbb{N}^*;\lambda_k=\lambda_{k+1}\}.$$

For every $k\in\Lambda_2$, the vectors $\langle\mu\phi_1,\phi_k\rangle$ and $\langle\mu\phi_1,\phi_{k+1}\rangle$ are linearly independent in $\mathbb{R}^2$, thus there exists a unique $Z_k\in\mathbb{C}^2$ such that

$$\Big\langle\langle\mu\phi_1,\phi_k\rangle_{L^2(\Omega)},Z_k\Big\rangle_{\mathbb{R}^2}=-id_k,\quad \Big\langle\langle\mu\phi_1,\phi_{k+1}\rangle_{L^2(\Omega)},Z_k\Big\rangle_{\mathbb{R}^2}=-id_{k+1},$$

where $d_j:=\langle\Psi_f,\phi_j\rangle e^{i\lambda_j T}-\langle\Psi_0,\phi_j\rangle$, for every $j\in\mathbb{N}^*$. For a function $v\in L^2((0,T),\mathbb{R}^2)$, (5.15) is satisfied in particular when

$$\begin{array}{l}\int_0^T v(t)e^{i\omega_k t}dt=-id_k\frac{\langle\mu\phi_1,\phi_k\rangle}{|\langle\mu\phi_1,\phi_k\rangle|^2},\forall k\in\Lambda_1,\\ \int_0^T v(t)e^{i\omega_k t}dt=Z_k,\forall k\in\Lambda_2,\end{array} \tag{5.21}$$

i.e. when $v^{(1)}$ and $v^{(2)}$ solve a trigonometric moment problem with a finite supported right hand side. The solvability of (5.21) is equivalent to the minimality of the family

$$\{1,\cos(\omega_k t),\sin(\omega_k t); k\ge 2\}$$

in $L^2((0,T),\mathbb{R})$ (see Proposition 5.9), which holds true thanks to Theorem 5.10.

For the proof of **(1.b)** and **(3)**, let us first emphasize that, when (5.18) and (Kal) hold, then the spectral controllability in time T of (5.3) is *equivalent to* (and not only implied by) the minimality of the family $\{1,\cos(\omega_k t),\sin(\omega_k t); k\ge 2\}$ in $L^2((0,T),\mathbb{R})$.

(1.b) Let $T < T_{min}$ and let us assume (5.18). Theorem 5.10 ensures that the family $\{1, \cos(\omega_k t), \sin(\omega_k t); k \geq 2\}$ is not minimal in $L^2((0,T),\mathbb{R})$, thus (5.4) is not spectral controllable in time T.

(2) Let Ω be a domain of $\mathbb{R}^n$ with C^1 boundary, $n = 2, 3$, and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$. We assume that (Kal) does not hold. There exists $k \in \mathbb{N}^*$ such that λ_k has multiplicity m and there exists $(\alpha_1, ..., \alpha_m) \in \mathbb{R}^m - \{0\}$ such that $\alpha_1 \langle \mu\phi_1, \phi_{k_1}\rangle + ... + \alpha_m \langle \mu\phi_1, \phi_{k_m}\rangle = 0$, where $k_1, ..., k_m$ are all the integers such that $\lambda_k = \lambda_{k_1} = ... = \lambda_{k_m}$. Let $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(T)$ of the form $\Psi_0 = \beta_1\phi_{k_1} + ... + \beta_m\phi_{k_m}$ where $\beta_1, ..., \beta_m \in \mathbb{C}$ and $\alpha_1\beta_1 + ... + \alpha_m\beta_m \neq 0$. Any solution of (5.6) satisfies, for $j \in \{1, ..., m\}$,

$$\langle \Psi(T), \phi_{k_j}\rangle = \Big(\langle \Psi_0, \phi_{k_j}\rangle + i\Big\langle \langle \mu\phi_1, \phi_{k_j}\rangle, \int_0^T v(t)e^{i\omega_k t}dt\Big\rangle_{\mathbb{R}^n}\Big)e^{-i\lambda_k T}.$$

We then have

$$\alpha_1\langle \Psi(T), \phi_{k_1}\rangle + ... + \alpha_m\langle \Psi(T), \phi_{k_m}\rangle = (\alpha_1\beta_1 + ... + \alpha_m\beta_m)e^{-i\lambda_k T} \neq 0,$$

implying that Ψ_0 is not zero controllable in time T.

(3) Let Ω be a domain of $\mathbb{R}^3$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^3)$ of the form (5.18) be such that (Kal) holds true (otherwise, we already know that (5.3) is not spectral controllable thanks to **(2)**). Let $T > 0$. Thanks to Weyl's formula, we have

$$\sharp\{\omega_k \in [0,t]\} = dt^{3/2} + O(t^\alpha), \text{ when } t \to +\infty,$$

where $d \in (0, +\infty)$ and $\alpha \in (0, 3/2)$. Thus, there exists a subsequence $(\omega_{\sigma(k)})_{k\in\mathbb{N}^*}$ of $(\omega_k)_{k\in\mathbb{N}^*}$ such that

$$\sharp\{k \in \mathbb{N}^*; \omega_{\sigma(k)} \in [0,t]\} = d't + O(t^{\alpha'}) \text{ when } t \to +\infty,$$

for some $d' > T/2\pi$ and some $\alpha' \in (0,1)$. Theorem 5.10 ensures that the family

$$\{e^{i\omega_{\sigma(k)}t}, e^{-i\omega_{\sigma(k)}t}; k \in \mathbb{N}^*\}$$

is not minimal in $L^2((0,T),\mathbb{C})$. Thus, the family $\{1, e^{i\omega_k t}, e^{-i\omega_k t}; k \geq 2\}$ is not minimal in $L^2((0,T),\mathbb{C})$. Therefore, (5.3) is not spectral controllable.

□

Remark 5.7. When a domain Ω of $\mathbb{R}^2$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ are such that (Kal) holds but (5.18) does not hold, then $T_{min}(\Omega) := 2\pi d(\Omega)$ may not be the minimal time for the spectral controllability of (5.3). Indeed, let us consider $\mu = (\mu^{(1)}, \mu^{(2)})$ such that

$$\langle \mu^{(1)}\phi_1, \phi_k\rangle \neq 0 \text{ if and only if } k \in \mathbb{N}^* \text{ is odd,}$$

$$\langle \mu^{(2)}\phi_1, \phi_k\rangle \neq 0 \text{ if and only if } k \in \mathbb{N}^* \text{ is even.}$$

Then, the minimal time for the spectral controllability of (5.3) is

$$T_{min}(\Omega, \mu) = \pi d(\Omega).$$

Remark 5.8. In order to remove the assumption (5.18), one could try to adapt Haraux and Jaffard's result to families of vector exponentials of the form

$$\{b_k e^{i\omega_k t}; k \in \mathbb{Z}\}$$

where $b_k \in \mathbb{R}^n - \{0\}$. Indeed, the spectral controllability of (5.3) is equivalent to the minimality in $L^2((0,T),\mathbb{C}^n)$ of this family with $b_k = \langle \mu\phi_1, \phi_k\rangle$. This generalization is an open problem.

5.3.3 Proof of Theorem 5.6

The goal of this section is the proof of Theorem 5.6. The proof of the statement **(1)** can be deduced from the following lemma in the same way as the proof of Theorem 5.5 **(1.a)** was deduced from Theorem 5.10 **(1)**.

Lemma 5.11. *Let $(\mu_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers such that $\mu_0 = 0$ and*

$$\sharp\{k \in \mathbb{Z}; 0 \leq \mu_k \leq t\} = dt + O(t^\alpha),$$
$$\sharp\{k \in \mathbb{Z}; -t \leq \mu_k \leq 0\} = dt + O(t^\alpha),$$

for some $d > 0$ and $\alpha \in (0,1)$. Then, for every $T > 2\pi d$, the family $\{t, e^{i\mu_k t}; k \in \mathbb{Z}\}$ is minimal in $L^2((0,T),\mathbb{C})$.

Proof of Lemma 5.11. Let $T > 2\pi d$ and let us assume that the family $\{t, e^{i\mu_k t}; k \in \mathbb{Z}\}$ is not minimal in $L^2((0,T),\mathbb{C})$. Thanks to Theorem 5.10, the family $\{e^{i\mu_k t}; k \in \mathbb{Z}\}$ is minimal in $L^2((0,T),\mathbb{C})$ thus, necessarily,

$$t \in \mathrm{Cl}_{L^2((0,T),\mathbb{C})}\Big(\mathrm{Span}\{e^{i\mu_k t}; k \in \mathbb{Z}\}\Big). \tag{5.22}$$

With successive integrations, we see that

$$t^k \in \mathrm{Cl}_{C^0([0,T],\mathbb{C})}\Big(\mathrm{Span}\{t, e^{i\mu_k t}; k \in \mathbb{Z}\}\Big), \forall k \in \mathbb{N}, k \geq 2.$$

The Stone Weirstrass theorem ensures that $\mathrm{Span}\{1, t^k; k \in \mathbb{N}, k \geq 2\}$ is dense in $C^0([0,T],\mathbb{C})$, thus it is also dense in $L^2((0,T),\mathbb{C})$. From (5.22), we deduce that the vector space $\mathrm{Span}\{e^{i\mu_k t}; k \in \mathbb{Z}\}$ is dense in $L^2((0,T),\mathbb{C})$. This is a contradiction, because, thanks to Theorem 5.10, for every $\alpha \in \mathbb{R} - \{\mu_k; k \in \mathbb{Z}\}$, the family $\{e^{i\alpha t}, e^{i\mu_k t}; k \in \mathbb{Z}\}$ is minimal in $L^2((0,T),\mathbb{C})$ i.e.

$$e^{i\alpha t} \notin \mathrm{Cl}_{L^2((0,T),\mathbb{C})}\Big(\mathrm{Span}\{e^{i\mu_k t}; k \in \mathbb{Z}\}\Big).$$

□

Item **(2)** of Theorem 5.6 is a direct consequence of Theorem 5.5 **(2)**. The proof of the statement **(3)** of Theorem 5.6 involves different ideas. A useful preliminary result is stated in the next Lemma, (see [65, Lecture 2, section 2.3, p.10-11]) that has already been used in [26, Lemma 16] for similar purposes.

Lemma 5.12. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that*

$$\exists C_0 > 0, \text{ such that }, \forall s \in \mathbb{C}, |f(s)| \leq C_0 e^{C_0|s|}.$$

Assume that $f \neq 0$. Let $n : [0,+\infty) \to \mathbb{N}$ be defined by

$$n(R) := \sharp\{s \in \mathbb{C}; f(s) = 0 \text{ and } |s| \leq R\}.$$

Then,

$$\exists C_1 > 0, \forall R \in (1,+\infty), \int_1^R \frac{n(t)}{t} dt \leq C_1 R.$$

*Proof of Item **(3)** of Theorem 5.6.* Let Ω be a regular domain of $\mathbb{R}^3$ such that (Kal) holds (otherwise, the system (5.4) is already known to be non spectral controllable thanks to **(2)**). Let $T>0$ and $m\in\mathbb{N}^*$. We assume $(i\phi_m,0,e_l)$ is zero controllable in time T for $l=1,2,3$: there exists $v_l\in L^2((0,T),\mathbb{R}^3)$ such that

$$\left\{\begin{array}{l}\left\langle\langle\mu\phi_1,\phi_k\rangle_{L^2(\Omega)},\int_0^T v_l(t)e^{i\omega_k t}dt\right\rangle_{\mathbb{R}^3}=-\delta_{k,m},\forall k\geq 2,\\ \int_0^T v_l(t)dt=0,\\ \int_0^T tv_l(t)dt=e_l,\end{array}\right. \tag{5.23}$$

for $l=1,2,3$. In particular, for every $k\in\mathbb{N}^*-\{1,m\}$, the vector $\langle\mu\phi_1,\phi_k\rangle_{L^2(\Omega)}\in\mathbb{R}^3-\{0\}$ belongs to the kernel of the matrix $C(i\omega_k)$, where

$$C(\lambda):=\left(\begin{array}{ccc}\int_0^T v_1^{(1)}(t)e^{\lambda t}dt & \int_0^T v_1^{(2)}(t)e^{\lambda t}dt & \int_0^T v_1^{(3)}(t)e^{\lambda t}dt\\ \int_0^T v_2^{(1)}(t)e^{\lambda t}dt & \int_0^T v_2^{(2)}(t)e^{\lambda t}dt & \int_0^T v_2^{(3)}(t)e^{\lambda t}dt\\ \int_0^T v_3^{(1)}(t)e^{\lambda t}dt & \int_0^T v_3^{(2)}(t)e^{\lambda t}dt & \int_0^T v_3^{(3)}(t)e^{\lambda t}dt\end{array}\right).$$

Thus $G(\lambda):=\det[C(\lambda)]$ satisfies $G(i\omega_k)=0$, for every $k\in\mathbb{N}^*-\{1,m\}$. It is easy to see that G is a holomorphic function verifying the growth condition of Lemma 5.12. Then using Weyl's formula and Lemma 5.12, we deduce that $G\equiv 0$. However, thanks to the last two equalities in (5.23), we have

$$C(\lambda)=\left(\begin{array}{ccc}\lambda+o(\lambda) & o(\lambda) & o(\lambda)\\ o(\lambda) & \lambda+o(\lambda) & o(\lambda)\\ o(\lambda) & o(\lambda) & \lambda+o(\lambda)\end{array}\right)\text{ when }\lambda\to 0,$$

so $G(\lambda)=\lambda^3+o(\lambda^3)\neq 0$ when $\lambda\to 0$, which is a contradiction.

□

5.4 2D exact controllability in abstract spaces

The goal of this section is the proof of the following result.

Theorem 5.13. *Let Ω be a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu\in C^0(\overline{\Omega},\mathbb{R}^2)$ be of the form (5.18) such that condition (Kal) holds true. Let $d\in(0,+\infty)$ and $\alpha\in(0,1)$ be such that (5.20) holds, $T>2\pi d$ and $(x_m)_{m\in\mathbb{N}^*}\subset\mathbb{R}_+^*$ be such that $\sum_{m=1}^\infty x_m=1$.*

For every $m\in\mathbb{N}^*$, there exists $C_m>0$ such that, for every $\varphi_T\in T_S\psi_1(T)$, the solution of

$$\left\{\begin{array}{ll}i\frac{\partial\varphi}{\partial t}=-\Delta\varphi, & (t,q)\in\mathbb{R}_+\times\Omega,\\ \varphi(t,q)=0, & (t,q)\in\mathbb{R}\times\partial\Omega,\\ \varphi(T)=\varphi_T,\end{array}\right. \tag{5.24}$$

satisfies

$$C_m|\langle\varphi_T,\phi_m\rangle|^2\leq\int_0^T|\Im\langle\tilde{\mu}\psi_1(t),\varphi(t)\rangle|^2dt,\forall m\in\mathbb{N}^*. \tag{5.25}$$

We introduce the Hilbert spaces

$$H^*:=\{\varphi:\Omega\to\mathbb{C};\Re\langle\varphi,\psi_1(T)\rangle=0\text{ and }\sum_{m=1}^\infty C_m x_m|\langle\varphi,\phi_m\rangle|^2<+\infty\},$$
$$H:=\{\varphi:\Omega\to\mathbb{C};\Re\langle\varphi,\psi_1(T)\rangle=0\text{ and }\sum_{m=1}^\infty\tfrac{1}{C_m x_m}|\langle\varphi,\phi_m\rangle|^2<+\infty\}.$$

Then, for every $\Psi_f\in H$, there exists $\tilde{v}\in L^2((0,T),\mathbb{R})$ such that the solution of (5.6) with $\Psi_0=0$ and $v(t)=\tilde{v}(t)e_1$ satisfies $\Psi(T)=\Psi_f$.

Remark 5.9. Notice that $T_S\psi_1(T) \subset H^*$ and $H \subset T_S\psi_1(T)$ because $C_m x_m \to 0$ when $m \to +\infty$. The space H is a regular space, its regularity depends on the asymptotic behavior of the sequence $(C_m x_m)_{m\in\mathbb{N}^*}$.

Remark 5.10. The spaces H and H^* are defined in order to have an observability inequality in H^*. Indeed, considering the product of the inequality (5.25) with x_m and summing over $m \in \mathbb{N}^*$, we get

$$\|\varphi_T\|^2_{H^*} \leq \int_0^T |\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle|^2 dt \ , \ \forall \varphi_T \in H^*. \tag{5.26}$$

Remark 5.11. Trying to apply the classical approach in order to get the controllability thanks to (5.26), we introduce the functional

$$\begin{array}{rccl} J: & H^* & \to & \mathbb{R} \\ & \varphi_T & \mapsto & \frac{1}{2}\int_0^T |\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle|^2 dt + \Re\langle\varphi_T, \Psi_f\rangle. \end{array}$$

In the classical situation, J is continuous, convex and coercive on H^*, thus $\inf\{J(\varphi_T); \varphi_T \in H^*\}$ is achieved at some point φ_T. Writing $dJ(\varphi_T) = 0$, we get a control $\tilde{v}(t) := \Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle$ that steers (5.3) from $\Psi(0) = 0$ to $\Psi(T) = \Psi_f$.

In our situation, this classical approach does not work because the functional J may not be well defined on H^*. Thus, an adaptation of this approach is needed.

Proof of Theorem 5.13. First, let us prove (5.25). For $\varphi_T \in T_S\psi_1(T)$, the solution of (5.24) is

$$\varphi(t) = \sum_{k=1}^{\infty} \langle\varphi_T, \phi_k\rangle e^{-i\lambda_k(t-T)}\phi_k$$

so

$$\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle = \sum_{k=2}^{\infty} \frac{\langle\tilde{\mu}\phi_1, \phi_k\rangle}{2i}\Big(\overline{\langle\varphi_T, \phi_k\rangle} e^{i\lambda_k(t-T)} - \langle\varphi_T, \phi_k\rangle e^{-i\lambda_k(t-T)}\Big).$$

Applying Theorem 5.10 and Proposition 5.9, there exists a constant $\tilde{C}_m > 0$ such that, for every $\varphi_T \in T_S\psi_1(T)$,

$$\tilde{C}_m |\langle\tilde{\mu}\phi_1, \phi_m\rangle|^2 |\langle\varphi_T, \phi_m\rangle|^2 \leq \int_0^T |\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle|^2 dt.$$

We get (5.25) with $C_m := \tilde{C}_m |\langle\tilde{\mu}\phi_1, \phi_m\rangle|^2$.

Now, let us prove the controllability result. Let $\Psi_f \in H$. For $\epsilon > 0$ we introduce the functional $J_\epsilon : T_S\psi_1(T) \to \mathbb{R}$,

$$J_\epsilon(\varphi_T) := \frac{1}{2}\int_0^T |\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle|^2 dt + \Re\langle\Psi_f, \varphi_T\rangle + \epsilon\|\varphi_T\|^2_{L^2(\Omega)},$$

where φ is the solution of (5.24). The functional J_ϵ is convex, continuous and coercive because

$$J_\epsilon(\varphi_T) \geq \epsilon\|\varphi_T\|^2_{L^2} - \|\Psi_f\|_{L^2}\|\varphi_T\|_{L^2}.$$

Thus, there exists $\varphi_T^\epsilon \in T_S\psi_1(T)$ such that

$$J_\epsilon(\varphi_T^\epsilon) = \min\{J_\epsilon(\varphi_T); \varphi_T \in T_S\psi_1(T)\}.$$

Then, φ_T^ϵ solves the Euler equation associated to this optimization problem,

$$\int_0^T \tilde{v}_\epsilon(t)\Im\langle\tilde{\mu}\psi_1(t),\xi(t)\rangle dt + \Re\langle\Psi_f,\xi_T\rangle + 2\epsilon\Re\langle\varphi_T^\epsilon,\xi_T\rangle = 0, \forall\xi_T \in T_S\psi_1(T), \tag{5.27}$$

where

$$\tilde{v}_\epsilon(t) := \Im\langle\tilde{\mu}\psi_1(t),\varphi^\epsilon(t)\rangle,$$

φ^ϵ (resp. ξ) is the solution of (5.24) with $\varphi_T = \varphi_T^\epsilon$ (resp. $\varphi_T = \xi_T$).

For $0 < \epsilon_1 < \epsilon_2$, we have $J_{\epsilon_1} \leq J_{\epsilon_2}$ thus the sequence $(J_\epsilon(\varphi_T^\epsilon))_{\epsilon>0}$ decreases when ϵ decreases to zero. Thus,

$$J_\epsilon(\varphi_T^\epsilon) \leq M_1 := J_1(\varphi_T^1), \forall\epsilon \in (0,1).$$

There exists $M_2 > 0$ such that,

$$\|\varphi_T^\epsilon\|_{H^*} \leq M_2, \forall\epsilon \in (0,1).$$

Indeed, thanks to (5.26), we have,

$$M_1 \geq J_\epsilon(\varphi_T^\epsilon) \geq \frac{1}{2}\|\varphi_T^\epsilon\|_{H^*}^2 - \|\Psi_f\|_H\|\varphi_T^\epsilon\|_{H^*}.$$

The sequence $(\tilde{v}_\epsilon)_{\epsilon\in(0,1)}$ is bounded in $L^2((0,T),\mathbb{R})$. Indeed, we have

$$M_1 \geq J_\epsilon(\varphi_T^\epsilon) \geq \frac{1}{2}\|\tilde{v}_\epsilon\|_{L^2}^2 - \|\Psi_f\|_H\|\varphi_T^\epsilon\|_{H^*},$$

thus

$$\|\tilde{v}_\epsilon\|_{L^2(0,T)}^2 \leq 2(M_1 + M_2\|\Psi_f\|_H).$$

Therefore, there exists $\tilde{v} \in L^2((0,T),\mathbb{R})$ such that $\tilde{v}_\epsilon \to \tilde{v}$ weakly in $L^2((0,T),\mathbb{R})$. Passing to the limit $\epsilon \to 0$ in (5.27) with $\xi_T \in H$, we get

$$\int_0^T \tilde{v}(t)\Im\langle\tilde{\mu}\psi_1(t),\xi(t)\rangle dt + \Re\langle\Psi_f,\xi_T\rangle = 0, \forall\xi_T \in H,$$

because

$$|2\epsilon\Re\langle\varphi_T^\epsilon,\xi_T\rangle| \leq 2\epsilon\|\varphi_T^\epsilon\|_{H^*}\|\xi_T\|_H \leq 2\epsilon M_2\|\xi_T\|_H.$$

Since H is dense in $T_S\psi_1(T)$, we have

$$\int_0^T \tilde{v}(t)\Im\langle\tilde{\mu}\psi_1(t),\xi(t)\rangle dt + \Re\langle\Psi_f,\xi_T\rangle = 0, \forall\xi_T \in T_S\psi_1(T). \tag{5.28}$$

Let Ψ be the solution of (5.6) with $\Psi_0 = 0$. Using the fact that ξ solves (5.24) and Ψ solves (5.6) with $\Psi_0 = 0$, we deduce from (5.28) that

$$\Re\langle\Psi(T),\xi_T\rangle = \Re\langle\Psi_f,\xi_T\rangle \ , \forall\xi_T \in T_S\psi_1(T).$$

Thus $\Psi(T) = \Psi_f$.

□

Remark 5.12. A uniform gap condition for the eigenvalues of $-\Delta_\Omega^D$, cf. (5.12), would imply that the constants C_m, $m \in \mathbb{N}^*$ admit a uniform positive lower bound and, in that case, H can be taken as the subset of $T_S\psi_1(T)$ made of the functions ϕ with $H^{1+\epsilon}$ finite norm. As we mentioned before, the existence of a planar domain verifying (5.12) is not even known. One could maybe define weaker gaps conditions in order to relate H to some Sobolev spaces.

5.5 Generic spectral controllability for the quantum box

The goal of this Section is the proof of Proposition 5.8.

Consider $\mu \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. If μ is not nowhere constant, then there exists an open ball B where μ is constant. Taking an open neighborhood of domains of $\mathbb{D}_3$ included in B, condition $(NonZ)$ will never be satisfied for those domains, thus Property (A) is not generic in $\mathbb{D}_3$.

For the rest of the section, we fix $\mu \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ which is nowhere constant.

In Subsection 5.5.1, we reduce the proof of the genericity of Property (A) (Proposition 5.8) to the proof of the genericity of a weaker property (B_k). In Subsection 5.5.2, we present the strategy for the proof of the genericity of Property (B_k) : it is sufficient to prove a weaker result, stated in Proposition 5.17. In subsection 5.5.3, we present the strategy for the proof of Proposition 5.17. In Subsection 5.5.4, we perform some preliminary results for the proof of Proposition 5.17, which is achieved in Subsection 5.5.5.

5.5.1 Reduction of the problem

The goal of this section is to reduce the proof of the genericity of the property (A), (Proposition 5.8) to the proof of the genericity of a weaker property (B_k). For that purpose, we introduce the properties (A_k) and (B_k).

For the rest of the paper, the notations $\lambda_j^{\Omega_0}$ and $\phi_j^{\Omega_0}$ are used to denote respectively the jth eigenvalue and one corresponding normalized eigenvector associated to $-\Delta_{\Omega_0}^D$. If, in the course of a definition or an argument, one domain under consideration is denoted Ω, then we simply use λ_j and ϕ_j instead of λ_j^{Ω} and ϕ_j^{Ω}.

Définition 5.10. Let $k \in \mathbb{N}^*$, $k \geq 2$ and $\Omega \in \mathbb{D}_3$. We say that Ω satisfies Property (A_k) if

$$\int_\Omega \mu(q)\phi_1(q)\phi_k(q)dq \neq 0.$$

Définition 5.11. Let $k \in \mathbb{N}^*$, $k \geq 2$ and $\Omega \in \mathbb{D}_3$. We say that Ω satisfies Property (B_k) if

either

$$\int_\Omega \mu(q)\phi_1(q)\phi_k(q)dq \neq 0,$$

or

$$\int_\Omega \mu(q)\phi_1(q)\phi_k(q)dq = 0 \text{ and } M(\cdot) \text{ is not identically equal to zero,} \tag{5.29}$$

where $M : \partial\Omega \to \mathbb{R}^2$ is given by

$$M(q) := \frac{\partial\phi_1}{\partial\nu}(q)\frac{\partial\xi_k}{\partial\nu}(q) + \frac{\partial\phi_k}{\partial\nu}(q)\frac{\partial\xi_1}{\partial\nu}(q), \tag{5.30}$$

ν is the unit outward normal to $\partial\Omega$ and ξ_1, ξ_k are the solutions of the following systems,

$$\begin{cases} -(\Delta+\lambda_1)\xi_k = \mu\phi_k, & \text{in } \Omega, \\ \xi_k = 0, & \text{on } \partial\Omega, \\ \int_\Omega \xi_k\phi_1 = 0, \end{cases} \quad \begin{cases} -(\Delta+\lambda_k)\xi_1 = \mu\phi_1, & \text{in } \Omega, \\ \xi_1 = 0, & \text{on } \partial\Omega, \\ \int_\Omega \xi_1\phi_k = 0. \end{cases} \tag{5.31}$$

A first reduction is given in the next proposition. Its proof is standard and relies on Baire Lemma, we write it for sake of completeness.

Proposition 5.14. *If (A_k) is generic in $\mathbb{D}_3$ for every $k \geq 2$, then (A) is generic in $\mathbb{D}_3$.*

Proof of Proposition 5.14. Let $\Omega \in \mathbb{D}_3$. We want to prove that the set

$$\mathcal{G} := \{u \in W^{4,\infty}(\Omega, \mathbb{R}^2); \Omega + u \text{ satisfies } (A)\}$$

is dense in $W^{4,\infty}(\Omega, \mathbb{R}^2)$. For $k \in \mathbb{N}^*$, we introduce the set $\mathcal{G}_k$ of functions $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ such that

$$\begin{aligned} &\lambda_1^{\Omega+u} < ... < \lambda_k^{\Omega+u} \leq \lambda_{k+1}^{\Omega+u} \leq ... \\ \text{and} \quad &\int_{\Omega+u} \mu(q)\phi_1^{\Omega+u}(q)\phi_j^{\Omega+u}(q)dq \neq 0, \forall j \in \{2, ..., k\}. \end{aligned}$$

Then, $\mathcal{G}_1 = W^{4,\infty}(\Omega, \mathbb{R}^2)$, $\mathcal{G}_{k+1}$ is an open subset of $\mathcal{G}_k$ for every $k \in \mathbb{N}^*$ (thanks to the continuity of $u \mapsto \lambda_j^{\Omega+u}$ and $u \mapsto \phi_j^{\Omega+u}$ for $j = 2, ..., k+1$) and $\mathcal{G} = \cap_{k\in\mathbb{N}^*}\mathcal{G}_k$. Thanks to Baire Lemma, it is sufficient to prove that, for every $k \in \mathbb{N}^*$, $\mathcal{G}_{k+1}$ is a dense in $\mathcal{G}_k$.

Let $k \in \mathbb{N}^*$, $u_0 \in \mathcal{G}_k - \mathcal{G}_{k+1}$ and $\epsilon > 0$. We have

$$\begin{aligned} &\lambda_1^{\Omega_0} < ... < \lambda_k^{\Omega_0} \leq \lambda_{k+1}^{\Omega_0} \leq ..., \\ &\int_{\Omega_0} \mu(q)\phi_1^{\Omega_0}(q)\phi_j^{\Omega_0}(q)dq \neq 0, \forall j \in \{2, ..., k\}, \end{aligned}$$

and $\lambda_k^{\Omega_0} = \lambda_{k+1}^{\Omega_0}$ or

$$\int_{\Omega_0} \mu(q)\phi_1^{\Omega_0}(q)\phi_{k+1}^{\Omega_0}(q)dq = 0,$$

where $\Omega_0 := \Omega + u_0$. Thanks to the generic simplicity of the eigenvalues of the Laplacian and the continuity of $u \mapsto \phi_j^{\Omega_0+u}$ for $2 \leq j \leq k$ (see [45]), there exists $u_1 \in W^{4,\infty}(\Omega_0, \mathbb{R}^2)$ with $\|u_1\|_{W^{4,\infty}} < \epsilon$ such that

$$\begin{aligned} &\lambda_1^{\Omega_1} < ... < \lambda_k^{\Omega_1} < \lambda_{k+1}^{\Omega_1} \leq ... \\ \text{and} \quad &\int_{\Omega_1} \mu(q)\phi_1^{\Omega_1}(q)\phi_j^{\Omega_1}(q)dq \neq 0, \forall j \in \{2, ..., k\}, \end{aligned}$$

where $\Omega_1 := \Omega_0 + u_1$. Thanks to the genericity of (A_{k+1}) and the continuity of $u \mapsto \lambda_j^{\Omega_1+u}$ for $2 \leq j \leq k+1$, $u \mapsto \phi_j^{\Omega_1+u}$ for $2 \leq j \leq k$ there exists $u_2 \in W^{4,\infty}(\Omega_1, \mathbb{R}^2)$ with $\|u_2\|_{W^{4,\infty}} < \epsilon$, such that

$$\begin{aligned} &\lambda_1^{\Omega_2} < ... < \lambda_k^{\Omega_2} < \lambda_{k+1}^{\Omega_2} \leq ... \\ \text{and} \quad &\int_{\Omega_2} \mu(q)\phi_1^{\Omega_2}(q)\phi_j^{\Omega_2}(q)dq \neq 0, \forall j \in \{2, ..., k+1\}, \end{aligned}$$

Then, $u := (I + u_2) \circ (I + u_1) \circ (I + u_0) - I$ is arbitrarily close to u_0 in $W^{4,\infty}(\Omega, \mathbb{R}^2)$ and $u \in \mathcal{G}_{k+1}$.

□

A second reduction is given in the next proposition. Its proof is also standard. The argument goes by contradiction and relies on shape differentiation with respect to the domain Ω. It has been introduced by Albert [3] and recently used in [26]. We gathered in Appendix 5.7 well-known facts about shape differentiation which will be used in the proof.

Proposition 5.15. *Let $k \geq 2$. If (B_k) is generic in $\mathbb{D}_3$, then (A_k) is generic in $\mathbb{D}_3$.*

Proof of Proposition 5.15. Let $\Omega_0 \in \mathbb{D}_3$, $k \in \mathbb{N}^*$, $k \geq 2$. We want to prove that the set

$$\mathcal{G} := \{u \in W^{4,\infty}(\Omega_0, \mathbb{R}^2); \Omega_0 + u \text{ satisfies } (A_k)\}$$

is dense in $W^{4,\infty}(\Omega_0, \mathbb{R}^2)$. We argue by contradiction. Let us assume the existence of $u_0 \in W^{4,\infty}(\Omega_0, \mathbb{R}^2)$ and $\rho_0 > 0$ such that, for every $u \in W^{4,\infty}(\Omega_0, \mathbb{R}^2)$ with $\|u_0 - u\|_{W^{4,\infty}} < \rho_0$, we have $u \notin \mathcal{G}$. Thanks to the genericity of (B_k), we can assume that $\Omega := \Omega_0 + u_0$ satisfies (B_k). Then, there exists $\rho > 0$ such that, for every $u \in E_\rho(\Omega) := \{v \in W^{4,\infty}(\Omega, \mathbb{R}^2); \|v\|_{W^{4,\infty}} < \rho\}$, we have

$$\int_{\Omega+u} \mu(q)\phi_1^{\Omega+u}(q)\phi_k^{\Omega+u}(q)dq = 0, \forall u \in E_\rho(\Omega). \tag{5.32}$$

Thus, the directional derivative of the integral appearing in (5.32) in the direction u is equal to zero, for every $u \in E_\rho(\Omega)$. By classical results on shape differentiation (cf. [89] or Appendix A below), we get

$$\int_\Omega \mu\Big(\phi_1'(u)\phi_k + \phi_1\phi_k'(u)\Big)dq = 0, \forall u \in E_\rho(\Omega), \tag{5.33}$$

where $\phi_1'(u)$ et $\phi_k'(u)$ are solutions of

$$\begin{cases} -(\Delta + \lambda_1)\phi_1'(u) = \lambda_1'(u)\phi_1, & \text{in } \Omega, \\ \phi_1'(u) = -\langle u, \nabla\phi_1\rangle, & \text{on } \partial\Omega, \\ \int_\Omega \phi_1\phi_1'(u) = 0, \end{cases}$$

$$\begin{cases} -(\Delta + \lambda_k)\phi_k'(u) = \lambda_k'(u)\phi_k, & \text{in } \Omega, \\ \phi_k'(u) = -\langle u, \nabla\phi_k\rangle, & \text{on } \partial\Omega, \\ \int_\Omega \phi_k\phi_k'(u) = 0. \end{cases} \tag{5.34}$$

In order to transform (5.33) into a linear form in u, we introduce the dual systems (5.31). Note that these systems have unique solutions, thanks to (5.32). Using Green's second formula and systems (5.34), we have

$$\begin{aligned} & -\int_\Omega \mu\Big(\phi_1'(u)\phi_k + \phi_1\phi_k'(u)\Big)dq \\ = & \int_\Omega \phi_1'(u)(\Delta + \lambda_1)\xi_k dq + \int_\Omega \phi_k'(u)(\Delta + \lambda_k)\xi_1 dq \\ = & \int_\Omega (\Delta + \lambda_1)\phi_1'(u)\xi_k dq + \int_{\partial\Omega} \Big(\phi_1'(u)\frac{\partial\xi_k}{\partial\nu} - \xi_k\frac{\partial\phi_1'(u)}{\partial\nu}\Big)d\sigma(q) \\ & + \int_\Omega (\Delta + \lambda_k)\phi_k'(u)\xi_1 dq + \int_{\partial\Omega} \Big(\phi_k'(u)\frac{\partial\xi_1}{\partial\nu} - \xi_1\frac{\partial\phi_k'(u)}{\partial\nu}\Big)d\sigma(q) \\ = & \int_{\partial\Omega} \Big(\phi_1'(u)\frac{\partial\xi_k}{\partial\nu} + \phi_k'(u)\frac{\partial\xi_1}{\partial\nu}\Big)d\sigma(q). \end{aligned}$$

Then, (5.33) is equivalent to

$$\int_{\partial\Omega} \langle u, \nu\rangle \left(\frac{\partial\phi_1}{\partial\nu}\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\frac{\partial\xi_1}{\partial\nu}\right) d\sigma(q) = 0, \forall u \in E_\rho(\Omega). \tag{5.35}$$

This implies that $M \equiv 0$ which is a contradiction because Ω satisfies (B_k).

□

5.5.2 Proof strategy for the genericity of (B_k)

According to Propositions 5.14 and 5.15, it remains to show that the Property (B_k) is generic in $\mathbb{D}_3$ for every $k \geq 2$. To proceed in that direction, fix $k \geq 2$ and $\Omega \in \mathbb{D}_3$. Without loss of generality, we assume from now that

1. the spectrum of $-\Delta_D$ is simple on Ω;
2. there exists $\bar{q} \in \partial\Omega$ such that
$$d\mu(\bar{q}) \cdot \tau_{\bar{q}} \neq 0, \tag{5.36}$$
where $\tau_{\bar{q}}$ is the unit tangent vector on $\partial\Omega$ at the point $\bar{q}$.

Indeed, the second condition is generic and open. Therefore, for a given domain $\Omega \in \mathbb{D}_3$, one can choose an arbitrarily close domain $\Omega' \in \mathbb{D}_3$ verifying condition 2. The latter holding in an open neighborhood of Ω', one can pick a domain $\Omega'' \in \mathbb{D}_3$ arbitrarily close to Ω veryfing both conditions 1 and 2.

Arguing by contradiction, we assume there exists $\rho > 0$ such that

$$\int_{\Omega+u} \mu(q)\phi_1^{\Omega+u}(q)\phi_k^{\Omega+u}(q)dq = 0, \forall u \in E_\rho(\Omega), \tag{5.37}$$

and

$$M(u) \equiv 0 \text{ on } \partial\Omega + u, \forall u \in E_\rho(\Omega), \tag{5.38}$$

where $E_\rho(\Omega) := \{v \in W^{4,\infty}(\Omega, \mathbb{R}^2); \|v\|_{W^{4,\infty}} < \rho\}$ and $M(u) : \partial(\Omega + u) \to \mathbb{R}^2$ is defined by

$$M(u)(q) = \frac{\partial\phi_k(u)}{\partial\nu}(q)\frac{\partial\xi_1(u)}{\partial\nu}(q) + \frac{\partial\phi_1(u)}{\partial\nu}(q)\frac{\partial\xi_k(u)}{\partial\nu}(q), \tag{5.39}$$

where $\phi_1(u)$, and $\phi_k(u)$ are normalized eigenvectors of $\Delta^D_{\Omega+u}$ associated to $\lambda_1(u)$ and $\lambda_k(u)$ respectively and $\xi_1(u)$ and $\xi_k(u)$ are the solutions of (5.31) associated to $\Omega+u$. (Such systems have solutions since (5.37) holds true.) In the sequel, we (sometimes) drop the variable (u) when it corresponds to $u = 0$.

The next step consists in shape differentiating the condition $M(u) \equiv 0$ for $u \in E_\rho(\Omega)$. Applying the classical shape differentiation formula regarding Dirichlet boundary condition (see Theorem A.2), we get

$$M'(u) = -\langle u, \nu\rangle \frac{\partial M(0)}{\partial\nu} \text{ on } \partial\Omega. \tag{5.40}$$

Remark 5.13. For technical details on regular extension of outward normal vector, we refer to [89, Théorème 4.1, Chapitre IV, page 69].

After computations, we get

$$\begin{aligned} &\left(\frac{\partial\phi_k}{\partial\nu}\right)'(u)\frac{\partial\xi_1}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\left(\frac{\partial\xi_1}{\partial\nu}\right)'(u) + \left(\frac{\partial\phi_1}{\partial\nu}\right)'(u)\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\left(\frac{\partial\xi_k}{\partial\nu}\right)'(u) \\ = \; &-\langle u, \nu\rangle \left(\frac{\partial}{\partial\nu}\left(\frac{\partial\phi_k}{\partial\nu}\right)\frac{\partial\xi_1}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\frac{\partial}{\partial\nu}\left(\frac{\partial\xi_1}{\partial\nu}\right)\right. \\ &\left. + \frac{\partial}{\partial\nu}\left(\frac{\partial\phi_1}{\partial\nu}\right)\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\frac{\partial}{\partial\nu}\left(\frac{\partial\xi_k}{\partial\nu}\right)\right) \text{ on } \partial\Omega. \end{aligned} \tag{5.41}$$

The relation between the first shape derivative of a normal derivative $(\frac{\partial\phi}{\partial\nu})'(u)$ and the normal derivative of a first shape derivative $\frac{\partial\phi'}{\partial\nu}$ is given in [44, Théorème 5.5.2, formula (5.74) of page 205] and reads as follows.

Lemma 5.16. *With the notations above, We have*

$$\left(\frac{\partial\phi}{\partial\nu}\right)' = \frac{\partial\phi'}{\partial\nu} - \langle u,\nu\rangle\left(\frac{\partial}{\partial\nu}\left(\frac{\partial\phi}{\partial\nu}\right) - \frac{\partial^2\phi}{\partial\nu^2}\right) - \langle\nabla\phi, \nabla_\Gamma(\langle u,\nu\rangle)\rangle \quad \text{on} \quad \partial\Omega, \tag{5.42}$$

where ∇_Γ is the tangential gradient and $\frac{\partial^2\phi}{\partial\nu^2}$ is understood as the image of the second derivative of ϕ (a bilinear form) applied to (ν,ν).

Using the above lemma and the fact that the involved functions vanish on $\partial\Omega$, (5.41) is rewritten as follows

$$\begin{aligned} &\frac{\partial\phi_k'(u)}{\partial\nu}\frac{\partial\xi_1}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\frac{\partial\xi_1'(u)}{\partial\nu} + \frac{\partial\phi_1'(u)}{\partial\nu}\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\frac{\partial\xi_k'(u)}{\partial\nu} \\ = \;& -\langle u,\nu\rangle\left(\frac{\partial^2\phi_k}{\partial\nu^2}\frac{\partial\xi_1}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\frac{\partial^2\xi_1}{\partial\nu^2} + \frac{\partial^2\phi_1}{\partial\nu^2}\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\frac{\partial^2\xi_k}{\partial\nu^2}\right), \end{aligned} \tag{5.43}$$

where $\phi_1'(u)$ and $\phi_k'(u)$ are defined in (5.34) and $\xi_1'(u)$ and $\xi_k'(u)$ are solutions of

$$\begin{cases} -(\Delta+\lambda_1)\xi_k'(u) = \lambda_1'(u)\xi_k + \mu\phi_k'(u), & \text{in} \quad \Omega, \\ \xi_k'(u) = -\langle u,\nu\rangle\frac{\partial\xi_k}{\partial\nu}, & \text{on} \quad \partial\Omega, \\ \int_\Omega\left(\phi_1\xi_k'(u) + \phi_1'(u)\xi_k\right)dq = 0, \end{cases}$$

and

$$\begin{cases} -(\Delta+\lambda_k)\xi_1'(u) = \lambda_k'(u)\xi_1 + \mu\phi_1'(u), & \text{in} \quad \Omega, \\ \xi_1'(u) = -\langle u,\nu\rangle\frac{\partial\xi_1}{\partial\nu}, & \text{on} \quad \partial\Omega, \\ \int_\Omega\left(\phi_k\xi_1'(u) + \phi_k'(u)\xi_1\right)dq = 0. \end{cases} \tag{5.44}$$

As a consequence of the previous computations, the genericity of (B_k) in $\mathbb{D}_3$ results from the next proposition.

Proposition 5.17. *Let $k \geq 2$ and $\Omega \in \mathbb{D}_3$. Assume that (5.37) and (5.38) hold true. Then, there does not exist $\rho' > 0$ such that (5.34) and (5.44) admit solutions satisfying (5.43) for every $u \in E_{\rho'}(\Omega)$.*

Remark 5.14. Let $J(\Omega)$ be a smooth functional depending on the domain Ω and u a variation belonging to $W^{k,\infty}(\Omega,\mathbb{R}^2)$. As pointed out in [89], we have

$$J''(\Omega)(u,u) = (J')'(\Omega)(u,u) - J'(\Omega)(u\cdot\nabla u). \tag{5.45}$$

This equation says that $J''(\Omega)$, the second derivative with respect to the domain at the point Ω, applied to the function u is not in general equal to the first derivative of the function $J'(\Omega)(u)$ at the point Ω applied to u. The difference between them is equal to the first shape derivative of the function $J(\Omega)$ applied to $u\cdot\nabla u$. However, in our case, they are equal because the first shape derivative is equal to zero by assumption. Thus, (5.43) exactly corresponds to the second shape derivative of (5.37).

5.5.3 Proof strategy for Proposition 5.17

To prove Proposition 5.17, our strategy is similar to that developed in [26] and, in order to describe it, we first need information on the regularity of the solutions of (5.34) and (5.44). For that purpose, we consider the following standard definitions of Sobolev spaces

and distributions on Ω (cf. [67]). If m is a positive integer, we use $H^m(\Omega)$ to denote the Sobolev space of order m on Ω defined by

$$H^m(\Omega) := \{\Psi \ | D^\alpha \Psi \in L^2(\Omega),\ |\alpha| \leq m\},$$

where $D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, and $|\alpha| = \alpha_1 + \alpha_2$. Here the differential operators D^α are defined in the distributional sense on Ω, with $\mathcal{D}'(\Omega)$ the space of distributions on Ω being dual to $\mathcal{D}(\Omega)$, the set of smooth functions with compact support in Ω (cf. [67]). Let $\rho : \overline{\Omega} \to \mathbb{R}^+$ be a function of class $C^2(\bar{\Omega})$ equal to the distance function to $\partial\Omega$ ($\rho(x) = d(x, \partial\Omega)$) for $d(x, \partial\Omega)$ small enough. Such a function exists as noted in [67, Chap.1, paragraph 11.2, page 62].

According to [67], for $s \in \mathbb{N}$, we set

$$\Xi^s(\Omega) := \{\Psi \mid \rho^{|\alpha|} D^\alpha \Psi \in L^2(\Omega),\ |\alpha| \leq s\},$$

equipped with the norm

$$\|\Psi\|_{\Xi^s(\Omega)} = \left(\sum_{|\alpha| \leq s} \|\rho^{|\alpha|} D^\alpha \Psi\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Then $\Xi^s(\Omega)$ is a Hilbert space so that $H^s(\Omega) \subset \Xi^s(\Omega) \subset \Xi^0(\Omega) = L^2(\Omega)$ with a continuous embedding. Let $\Xi^{-s}(\Omega) := (\Xi^s(\Omega))'$ be the dual space of $\Xi^s(\Omega)$ for the $L^2(\Omega)$ scalar-product. Then, $\Xi^{-s}(\Omega)$ is a distribution space as proved in [67].

Remark 5.15. By interpolations techniques, we can also define the spaces $\Xi^s(\Omega)$ for all real positive number s. Then, we have $H^s(\Omega) \subset \Xi^s(\Omega) \subset \Xi^{s'}(\Omega) \subset L^2(\Omega)$ if $0 < s' < s$ (see [67, p. 184] for more details).

We can now apply the general theorems stated in [67] to the present situation. Let $A := \Delta + \lambda$ and B_0 be the Dirichlet trace operator. We set

$$D_A^s(\Omega) = \{\Psi \ | \Psi \in H^s(\Omega), A\Psi \in \Xi^{s-2m}(\Omega)\}, \quad 0 < s < 2m,$$

with the norm defined by $\|\Psi\| = (\|\Psi\|^2_{H^s(\Omega)} + \|A\Psi\|^2_{\Xi^{s-2m}})^{1/2}$. Then, $D_A^s(\Omega)$ is a Hilbert space.

We write system (5.34) with new notations,

$$A\Psi = f \ \text{ in } \ \Omega \quad \text{and} \quad B_0\Psi = g \ \text{ on } \ \partial\Omega, \tag{5.46}$$

where

$$f = -\lambda'(u)\phi_j \text{ and } g = -\langle u, \nu \rangle \frac{\partial \phi_j}{\partial \nu}, \text{ with } j = 1, k. \tag{5.47}$$

We apply [67, Theorem 7.4 p. 202] for $m = 1$ (one boundary condition) and $m_0 = 0$ (there is not derivation in the trace operator). As ϕ_1 is an eigenfunction, f is in every distribution space, in particular it is an element of every $\Xi^{s'}(\Omega)$ for $s' < 0$. Then, if $0 < s < 2$, we have $f \in \Xi^{s-2}(\Omega)$. If $g \in H^{s-1/2}(\Omega)$, by [67, Theorem 7.4 p. 202], $\Psi \in D_A^s(\Omega)$. We now apply [67, Theorem 7.3 p.201] with $B_1 = \frac{\partial}{\partial \nu}$ and $m_1 = 1$. Then, we have $\frac{\partial \Psi}{\partial \nu} \in H^{s-3/2}(\Omega)$. We summarize these results in the following lemma.

Lemma 5.18. *Let $s \in (0, 2)$ and $j \in \{1, k\}$. With the notations above, if the Dirichlet boundary condition $g = \langle u, \nu \rangle \frac{\partial \phi_j}{\partial \nu} \in H^{s-1/2}(\partial\Omega)$, then we have $\phi_j'(u) \in H^s(\Omega)$ and $\frac{\partial \phi_j'(u)}{\partial \nu} \in H^{s-3/2}(\partial\Omega)$.*

As already mentioned in the introduction, the starting remark for the argument of Proposition 5.17 goes as follows. By taking into account Lemma 5.18, the right-hand side of (5.43) is in $H^{s-1/2}(\partial\Omega)$ and, at the same time, the left-hand side in $H^{s-3/2}(\partial\Omega)$, for $s \in (0,2)$. To take advantage of that gap of regularity between the two sides of (5.43), we first consider variations exhibiting just one jump of discontinuity on Ω, let say at some point $q_* \in \partial\Omega$, so that, for all the quantities involved in (5.43), an irregular part only occurs at the point q_*. If we are able to compute exactly this irregular part, we would infer that it has to be equal to zero by using (5.43). It would provide some extra information at the point q_*, of the type $F(q_*) = 0$ where F is an $\mathbb{R}^2$-valued map defined on $\partial\Omega$. Since the point q_* is arbitrary, we would end with the relation $F \equiv 0$ on $\partial\Omega$, similar to (5.38). Using this new information together with $M(0) \equiv 0$, one hopes to derive a contradiction.

Let us provide more details. Fix $q_* \in \partial\Omega$ and a parametrization σ of C_1, the connected component of $\partial\Omega$ containing q_*, so that $\sigma \in [-L, L)$ and q_* corresponds to $\sigma = 0$. Fix an open neighborhood V_α of q_* in C_1 parameterized by $(-\alpha, \alpha)$ with $\alpha < L$. We consider an admissible variation u_{q_*} (see Definition 5.12 below) defined as follows : on $(-\alpha, 0)$, $\langle u_{q_*}, \nu\rangle = 0$, on $(0, \alpha)$, $\langle u_{q_*}, \nu\rangle = 1$ and $\langle u_{q_*}, \nu\rangle$ is smooth in C_1 except at $\sigma = 0$. According to Remark 5.16 below, we can extend the definition of $M'(u)$ to functions u which are not regular enough to perform shape differentiation (such as u_{q_*}). We then show that $M'(u_{q_*})$ admits, in the distributional sense, the following Taylor expansion valid in $(-\alpha, \alpha)$,

$$M'(u_{q_*})(\sigma) = M_0 \text{p.v.}\left(\frac{1}{\sigma}\right) + M_1 \ln(|\sigma|) + M_3 \sigma \ln(|\sigma|) + \mathcal{R}(\sigma), \tag{5.48}$$

and we also have, according to (5.40),

$$M'(u_{q_*})(\sigma) = M_2 H_0(\sigma) + \mathcal{R}(\sigma).$$

In the above equations, the coefficients M_i, $0 \le i \le 3$, are $\mathbb{R}^2$-valued, $\mathcal{R}$ denotes a (generic) C^1 function over $(-\alpha, \alpha)$ and H_0 belongs to $H^{1/2-\epsilon}(\partial\Omega)$ for every $\epsilon > 0$.

We will then prove that M_i, $0 \le i \le 2$, are always equal to zero and, from the relation $M \equiv 0$ on $\partial\Omega$, we will therefore be left with the relation

$$M_3 \sigma \ln(|\sigma|) + \mathcal{R}(\sigma) = 0 \text{ on } (-\alpha, \alpha). \tag{5.49}$$

It would immediately yield $M_3 = 0$. Moreover, we will compute M_3 as a function of the values of ϕ_1, ϕ_k, ξ_1, ξ_k and their normal derivatives at $\sigma = 0$ (i.e., at q_*). Therefore, M_3 can be seen as a function defined on $\partial\Omega$ and, since q_* is arbitrary, we will get from (5.49) that $M_3(\cdot) \equiv 0$ on $\partial\Omega$. It will provide us with a new non trivial relationship between ϕ_1, ϕ_k, ξ_1, ξ_k and their normal derivatives and we will reach shortly after a contradiction, hence concluding the proof of Proposition 5.17.

In order now to access to (5.48) and get a hold on the M_i's, we split $M'(u_{q_*})$ as follows,

$$M'(u_{q_*}) = M'_b(u_{q_*}) + M'_d(u_{q_*}), \tag{5.50}$$

where

$$\begin{aligned} M'_b(u_{q_*}) &= \frac{\partial \phi'_k(u_{q_*})}{\partial \nu}\frac{\partial \xi_1}{\partial \nu} + \frac{\partial \phi_k}{\partial \nu}\frac{\partial \xi'_{1,b}(u_{q_*})}{\partial \nu} \\ &\quad + \frac{\partial \phi'_1(u_{q_*})}{\partial \nu}\frac{\partial \xi_k}{\partial \nu} + \frac{\partial \phi_1}{\partial \nu}\frac{\partial \xi'_{k,b}(u_{q_*})}{\partial \nu}, \end{aligned} \tag{5.51}$$

$$M'_d(u_{q_*}) = \frac{\partial \phi_k}{\partial \nu}\frac{\partial \xi'_{1,d}(u_{q_*})}{\partial \nu} + \frac{\partial \phi_1}{\partial \nu}\frac{\partial \xi'_{k,d}(u_{q_*})}{\partial \nu}, \tag{5.52}$$

where $M_b'(u_{q_*})$ and $M_d'(u_{q_*})$ are the contributions of respectively the boundary $\partial\Omega$ and the domain Ω to $M'(u_{q_*})$. In (5.51) and (5.52), we choose the variation u_{q_*} (see Definition 5.12) such that $\phi_1'(u_{q_*})$ and $\phi_k'(u_{q_*})$ are solutions of

$$\begin{cases} -(\Delta+\lambda_1)\phi_1'(u_{q_*}) = \lambda_1'(u_{q_*})\phi_1, & \text{in } \Omega, \\ \phi_1'(u_{q_*}) = -\langle u_{q_*},\nu\rangle \frac{\partial\phi_1}{\partial\nu}, & \text{on } \partial\Omega, \\ \int_\Omega \phi_1\phi_1'(u_{q_*}) = 0, \end{cases}$$

$$\begin{cases} -(\Delta+\lambda_k)\phi_k'(u_{q_*}) = \lambda_k'(u_{q_*})\phi_k, & \text{in } \Omega, \\ \phi_k'(u_{q_*}) = -\langle u_{q_*},\nu\rangle \frac{\partial\phi_k}{\partial\nu}, & \text{on } \partial\Omega, \\ \int_\Omega \phi_k\phi_k'(u_{q_*}) = 0, \end{cases} \tag{5.53}$$

and the $\xi_{1,b}'(u_{q_*})$, $\xi_{1,d}'(u_{q_*})$, $\xi_{k,b}'(u_{q_*})$ and $\xi_{k,d}'$ are defined as the solutions of the following Helmholtz equations,

$$\begin{cases} -(\Delta+\lambda_1)\xi_{k,b}'(u_{q_*}) = 0, & \text{in } \Omega, \\ \xi_{k,b}'(u_{q_*}) = -\langle u_{q_*},\nu\rangle \frac{\partial\xi_k}{\partial\nu}, & \text{on } \partial\Omega, \\ \int_\Omega \phi_1\xi_{k,b}'(u_{q_*}) = 0, \end{cases}$$

$$\begin{cases} -(\Delta+\lambda_k)\xi_{1,b}'(u_{q_*}) = 0, & \text{in } \Omega, \\ \xi_{1,b}'(u_{q_*}) = -\langle u_{q_*},\nu\rangle \frac{\partial\xi_1}{\partial\nu}, & \text{on } \partial\Omega, \\ \int_\Omega \phi_k\xi_{1,b}'(u_{q_*}) = 0, \end{cases} \tag{5.54}$$

$$\begin{cases} -(\Delta+\lambda_1)\xi_{k,d}'(u_{q_*}) = \lambda_1'(u)\xi_k + \mu\phi_k'(u_{q_*}), & \text{in } \Omega, \\ \xi_{k,d}'(u_{q_*}) = 0, & \text{on } \partial\Omega, \\ \int_\Omega \phi_1\xi_{k,d}'(u_{q_*}) = 0, \end{cases} \tag{5.55}$$

$$\begin{cases} -(\Delta+\lambda_k)\xi_{1,d}'(u_{q_*}) = \lambda_k'(u)\xi_1 + \mu\phi_1'(u_{q_*}), & \text{in } \Omega, \\ \xi_{1,d}'(u_{q_*}) = 0, & \text{on } \partial\Omega, \\ \int_\Omega \phi_k\xi_{1,d}'(u_{q_*}) = 0. \end{cases} \tag{5.56}$$

By linearity, we have

$$\xi_1'(u_{q_*}) = \xi_{1,b}'(u_{q_*}) + \xi_{1,d}'(u_{q_*}) + c_1\phi_k$$

and

$$\xi_k'(u_{q_*}) = \xi_{k,b}'(u_{q_*}) + \xi_{k,d}'(u_{q_*}) + c_2\phi_1,$$

where $c_1 = -\displaystyle\int_\Omega \phi_k'(u)\xi_1 dq$ and $c_2 = -\displaystyle\int_\Omega \phi_1'(u)\xi_k dq$. We simply intend here to compute $\xi_j'(u_{q_*})$, $j = 1, k$, as the sum of two terms, one coming from the boundary condition and the second from the inhomogeneous part of the PDE. Each of these terms requires the study of a Dirichlet-to-Neumann operator associated to a Helmholtz equation. In the next section, we develop in details these computations.

5.5.4 Evaluations of the singular parts of $M_b'(u_{q_*})$ and $M_d'(u_{q_*})$

In what follows, p and q denote points of $\mathbb{R}^2$ and x, y denotes respectively the first and second coordinates of a point in $\mathbb{R}^2$.

For the rest of the paper, we fix a point $q_* \in \partial\Omega$ and, with no loss of generality, we assume that $\partial\Omega$ has only one connected component.

We next choose a parametrization of $\partial\Omega$ by arc-length $\sigma \in [-L, L)$ so that q_* corresponds to $(x(0), y(0))$. The initial control problem (5.4) is clearly invariant by rotation and thus we can assume that the tangent vector at $\sigma = 0$ is equal to $(-1, 0)^T$. We finally proceed to a translation of vector q_* which implies that $(x(0), y(0)) = (0, 0)$. That transformation only modifies the PDEs governing ξ_j, $\xi'_{j,d}$ and $\xi'_{j,b}$, $j = 1, k$, replacing q by $q + q_*$ in (5.32), (5.31), (5.54) and (5.55).

Since Ω is of class C^3, there exists a neighborhood $\mathcal{N}_0$ of $0 \in \mathbb{R}$ such that for every $\sigma \in \mathcal{N}_0$, we have

$$x(\sigma) = -\sigma + O(\sigma^3), \tag{5.57}$$

$$y(\sigma) = \frac{\kappa(0)}{2}\sigma^2 + O(\sigma^3), \tag{5.58}$$

where κ is the curvature function of $\partial\Omega$. Let $\mathcal{N}_a$ be the subset of $\partial\Omega$ made of points $q(\sigma) = (x(\sigma), y(\sigma))$ with $\sigma \in \mathcal{N}_0$ and $\nu(\cdot)$ be the unit outward normal along $\partial\Omega$, which is of class C^2, and has direction $(y'(\cdot), -x'(\cdot))$.

We now consider a variation u_{q_*} which exhibits a unique jump of discontinuity at q_*, i.e. u_{q_*} is only defined through its normal part $\langle u_{q_*}, \nu\rangle$ given next

$$\langle u_{q_*}, \nu\rangle(\sigma) = \begin{cases} 0, & \sigma \in [-\alpha, 0), \\ 1, & \sigma \in [0, \alpha), \\ \eta(\sigma), & \sigma \in [-L, -\alpha) \cup [\alpha, L), \end{cases} \tag{5.59}$$

where $0 < \alpha$ is small enough so that $[-\alpha, \alpha] \subset \mathcal{N}_0$ and η is smooth so that $\langle u_{q_*}, \nu\rangle$ is $2L$-periodic and smooth except at $\sigma = 0$. We sometimes refer to $\langle u_{q_*}, \nu\rangle$ as the Heaviside function on $\partial\Omega$ and use $\mathcal{H}_0$ to denote it.

Remark 5.16. Strictly speaking, u_{q_*} cannot be considered as a variation of domain since it is not in $W^{4,\infty}(\Omega, \mathbb{R}^2)$. However, it is rather easy to see that solutions of the differential systems obtained after shape differentiation can be defined by standard density arguments for function spaces containing $W^{4,\infty}(\Omega, \mathbb{R}^2)$. For instance, $M'(u)$ is first defined by shape differentiation for $u \in E_\rho(\Omega)$, and that requires to consider the functions $\phi'_j(u)$ and $\xi'_j(u)$, $j = 1, k$ verifying (5.34) and (5.44). On the other hand, these functions only need $\langle u, \nu\rangle$, the normal component of the variation, to be defined. Thus, for $\langle u, \nu\rangle \in H^s(\partial\Omega)$, $s \leq 1$, one still can define by density (unique) solutions of (5.34) and (5.44) associated to u and thus traces on $\partial\Omega$ of these elements. Finally, using Lemma 5.18, the function defined in the left-hand side of (5.43) is well defined and, by an obvious abuse of notation, we use $M'(u)$ to denote it. We now have defined $M'(u_{q_*})$ and we refer to it as the shape differential of M for the variation u_{q_*}.

Remark 5.17. For presentation ease, we use the arc-length σ for parameterizing all points q in a neighborhood of the fixed point $q_* \in \partial\Omega$.

Définition 5.12. Let Ω be a domain of $\mathbb{D}_3$ not verifying condition (B_k). A variation u (defined with $\langle u, \nu\rangle \in H^s(\partial\Omega)$, $s \in (0, 2)$) is said to be *admissible* if

$$\int_{\partial\Omega} \langle u, \nu\rangle \frac{\partial\phi_1}{\partial\nu}\frac{\partial\xi_k}{\partial\nu} d\sigma(q) = 0. \tag{5.60}$$

By applying Green's second formula and using (5.33) and (5.38), one sees that condition (5.60) is necessary (and sufficient) for the existence of solutions of the PDEs given in (5.53), (5.54) and then (5.55) after an appropriate choice of c_1 and c_2. Moreover, remark that if $\frac{\partial\xi_k}{\partial\nu} \equiv 0$ on $\partial\Omega$ (and thus $\frac{\partial\xi_1}{\partial\nu} \equiv 0$), then every variation is admissible.

Lemma 5.19. *For every $q_* \in \partial\Omega$, one can choose the smooth function η and the parameter α introduced in (5.59) such that u_{q_*} is an admissible variation.*

Proof of Lemma 5.19. We may assume that $\frac{\partial \xi_k}{\partial \nu}$ (and thus $\frac{\partial \xi_1}{\partial \nu}$) is not identically equal to 0 on $\partial\Omega$. Assume first that $\frac{\partial \xi_k}{\partial \nu}(q_*) \neq 0$. Equation (5.60) can clearly be stated as an affine relation $L(\eta) = l$, where L is a linear form and $l \in \mathbb{R}$. Notice that L is not null. Indeed, $\frac{\partial \xi_k}{\partial \nu}(q)$ is not equal to zero in an open neighborhood of q_*. Then, by choosing α small enough, $\frac{\partial \xi_k}{\partial \nu}(q(\sigma)$ is not equal to zero for some σ in $(-L, -\alpha) \cup (\alpha, L)$. It is therefore always possible to select η so that u_{q_*} is an admissible variation. It is immediate to extend the above construction to the case where $\frac{\partial \xi_k}{\partial \nu}(q_*) = 0$ and there exists a sequence of points $q \in \partial\Omega$ converging to q_* such that $\frac{\partial \xi_k}{\partial \nu}(q) \neq 0$.

It remains to treat the case where $\frac{\partial \xi_k}{\partial \nu} \equiv 0$ on an open neighborhood $\mathcal{N}$ of $q_* \in \partial\Omega$. It is then possible to choose $\alpha > 0$ small enough so that $q(\sigma) \in \mathcal{N}$ for $\sigma \in (-2\alpha, 2\alpha)$ and $\eta \equiv 0$ on $(2\alpha, L) \cup (-L, -\alpha)$. Then, the corresponding u_{q_*} is admissible.

□

Définition 5.13. We say that a function g defined on $\partial\Omega$ is 2-regular if there exists two smooth (i.e., C^∞) functions $h, \tilde{h}$ defined on $\partial\Omega$ such that $g(\sigma) = \sigma^2 \ln(|\sigma|)h(\sigma) + \tilde{h}(\sigma)$ for σ in an open neighborhood of zero. We will use sometimes the symbol $\mathcal{R}_2$ to denote an arbitrary 2-regular function. In addition, we use the symbol $\mathcal{R}_1$ to denote an arbitrary C^1 function in an open neighborhood of zero. Note that a 2-regular function is necessarily of class C^1. Finally, we use the notation $O(\sigma)$ to denote an arbitrary C^1 function equal to zero at $\sigma = 0$ and with uniformly bounded derivative over some open neighborhood of zero.

In the next paragraph, we will prove that the irregular parts of $\frac{\partial \phi_1'}{\partial \nu}(u_{q_*})$, $\frac{\partial \phi_k'}{\partial \nu}(u_{q_*})$, $\frac{\partial \xi_{1,b}'}{\partial \nu}(u_{q_*})$, $\frac{\partial \xi_{1,d}'}{\partial \nu}(u_{q_*})$, $\frac{\partial \xi_{k,b}'}{\partial \nu}(u_{q_*})$ and $\frac{\partial \xi_{k,d}'}{\partial \nu}(u_{q_*})$ involved in $M'(u_{q_*}) = M_b'(u_{q_*}) + M_d'(u_{q_*})$ only occur at the point q_* and we intend to calculate them exactly.

5.5.4.1 Expression of $M_b'(u_{q_*})$

The main result of this section is the following theorem.

Theorem 5.20. *There exists an open neighborhood of zero $\mathcal{N}_1 \subset \mathcal{N}_0$ such that, if $\sigma \in \mathcal{N}_1$, one has*

$$M_b'(u_{q_*})(\sigma) = \frac{1}{\pi}\left\{\lambda_1 \frac{\partial \phi_1(0)}{\partial \nu}\frac{\partial \xi_k(0)}{\partial \nu} + \lambda_k \frac{\partial \phi_k(0)}{\partial \nu}\frac{\partial \xi_1(0)}{\partial \nu}\right\}\sigma \ln|\sigma| + \mathcal{R}_1. \tag{5.61}$$

For the rest of the paper, we set

$$a_1 := -\frac{1}{2\pi}, \quad a_2 := \frac{1}{8\pi}. \tag{5.62}$$

Note that the constant $1/\pi$ appearing in the right-hand side of (5.61) is equal to $-4(a_1 + 2a_2)$.

The proof of this theorem is based on the following proposition.

Proposition 5.21. *We have*

$$\begin{aligned}
\frac{\partial \phi_1'(u_{q_*})}{\partial \nu}(\sigma) &= -2\left\{a_1\frac{\partial \phi_1}{\partial \nu}(0)p.v.\left(\frac{1}{\sigma}\right)+a_1\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\ln|\sigma|\right.\\
&\quad\left.+\left\{a_1\frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \phi_1}{\partial \nu}\right)+(a_1+2a_2)\lambda_1\frac{\partial \phi_1}{\partial \nu}\right\}(0)\sigma\ln|\sigma|\right\}\\
&\quad -a_1\frac{\partial \phi_1}{\partial \nu}(0)L_1(\sigma)-a_1\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)L_2(\sigma)+\mathcal{R}_1, \qquad (5.63)\\
\frac{\partial \phi_k'(u_{q_*})}{\partial \nu}(\sigma) &= -2\left\{a_1\frac{\partial \phi_k}{\partial \nu}(0)p.v.\left(\frac{1}{\sigma}\right)+a_1\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_k}{\partial \nu}\right)(0)\ln|\sigma|\right.\\
&\quad\left.+\left\{a_1\frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \phi_k}{\partial \nu}\right)+(a_1+2a_2)\lambda_k\frac{\partial \phi_k}{\partial \nu}\right\}(0)\sigma\ln|\sigma|\right\}\\
&\quad -a_1\frac{\partial \phi_k}{\partial \nu}(0)L_1(\sigma)-a_1\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_k}{\partial \nu}\right)(0)L_2(\sigma)+\mathcal{R}_1, \qquad (5.64)\\
\frac{\partial \xi_{1,b}'(u_{q_*})}{\partial \nu}(\sigma) &= -2\left\{a_1\frac{\partial \xi_1}{\partial \nu}(0)p.v.\left(\frac{1}{\sigma}\right)+a_1\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0)\ln|\sigma|\right.\\
&\quad\left.+\left\{a_1\frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \xi_1}{\partial \nu}\right)+(a_1+2a_2)\lambda_k\frac{\partial \xi_1}{\partial \nu}\right\}(0)\sigma\ln|\sigma|\right\}\\
&\quad -a_1\frac{\partial \xi_1}{\partial \nu}(0)L_1(\sigma)-a_1\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0)L_2(\sigma)+\mathcal{R}_1, \qquad (5.65)\\
\frac{\partial \xi_{k,b}'(u_{q_*})}{\partial \nu}(\sigma) &= -2\left\{a_1\frac{\partial \xi_k}{\partial \nu}(0)p.v.\left(\frac{1}{\sigma}\right)+a_1\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_k}{\partial \nu}\right)(0)\ln|\sigma|\right.\\
&\quad\left.+\left\{a_1\frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \xi_k}{\partial \nu}\right)+(a_1+2a_2)\lambda_1\frac{\partial \xi_k}{\partial \nu}\right\}(0)\sigma\ln|\sigma|\right\}\\
&\quad -a_1\frac{\partial \xi_k}{\partial \nu}(0)L_1(\sigma)-a_1\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_k}{\partial \nu}\right)(0)L_2(\sigma)+\mathcal{R}_1, \qquad (5.66)
\end{aligned}$$

where $L_1(\sigma) := T_0(p.v.\left(\frac{1}{\sigma}\right))$, $L_2(\sigma) := T_0(\ln|\sigma|)$, *with* T_0 *the linear operator defined in (5.110). Recall that* $\mathcal{R}_1$ *is used to denote an arbitrary* C^1 *function of* $\partial\Omega$.

Proof of Proposition 5.21. Explicit computation is only provided for (5.63) since expressions for $\frac{\partial \phi_k'(u_{q_*})}{\partial \nu}$, $\frac{\partial \xi_{1,b}'(u_{q_*})}{\partial \nu}$ and $\frac{\partial \xi_{k,b}'(u_{q_*})}{\partial \nu}$ are derived in a similar way. From (5.106), we first easily get that the contribution of $\lambda_1'(u_{q_*})\phi_1$ to $\frac{\partial \phi_1'}{\partial \nu}(q_*)$ is a term of class C^2 and thus of type $\mathcal{R}_2$. We next apply Proposition 5.35 with $g = \frac{\partial \phi_1}{\partial \nu}$. It yields

$$\begin{aligned}
E_1\left(\mathcal{H}_0\frac{\partial \phi_1}{\partial \nu}\right)(\sigma) &= \frac{\partial \phi_1(0)}{\partial \nu}a_1\mathrm{p.v.}\left(\frac{1}{\sigma}\right)+\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)a_1\ln|\sigma|\\
&\quad+\left\{a_1\frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \phi_1}{\partial \nu}\right)+(a_1+2a_2)\lambda_1\frac{\partial \phi_1}{\partial \nu}\right\}(0)\sigma\ln|\sigma|\\
&\quad+\mathcal{R}_2. \qquad (5.67)
\end{aligned}$$

According to Theorem 5.31, we get

$$\begin{aligned}\frac{\partial\phi_1'}{\partial\nu}(\sigma) &= -2\left\{\frac{\partial\phi_1(0)}{\partial\nu}a_1\text{p.v.}\left(\frac{1}{\sigma}\right)+\frac{\partial}{\partial\tau}\left(\frac{\partial\phi_1}{\partial\nu}\right)(0)a_1\ln|\sigma|\right.\\ &\quad\left.+\left\{a_1\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial\phi_1}{\partial\nu}\right)+(a_1+2a_2)\lambda_1\frac{\partial\phi_1}{\partial\nu}\right\}(0)\sigma\ln|\sigma|\right\}\\ &\quad-\frac{\partial\phi_1(0)}{\partial\nu}a_1L_1(\sigma)-\frac{\partial}{\partial\tau}\left(\frac{\partial\phi_1}{\partial\nu}\right)(0)a_1L_2(\sigma)\\ &\quad-\left(a_1\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial\phi_1}{\partial\nu}\right)+(a_1+2a_2)\lambda_1\frac{\partial\phi_1}{\partial\nu}\right)(0)L_3(\sigma)+\mathcal{R}_1,\end{aligned}$$

where $L_3(\sigma):=T_0(\sigma\ln|\sigma|)$. Recalling that $\ln|\sigma|$ belongs to $H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$ and the regularizing effect of the operator T_0, one immediately gets that $\sigma\ln|\sigma|\in H^{3/2-\varepsilon}(\partial\Omega)$ and $L_3(\sigma)\in H^{5/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$. It implies that $L_3(\cdot)$ is a C^1 function of $\partial\Omega$.

□

Remark 5.18. For the rest of the paper, we will need information about the regularity of $L_j(\sigma)$, $j=1,2$. As done in the above argument, we have that p.v. $\left(\frac{1}{\sigma}\right)\in H^{-1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$ and, thanks to the regularizing effect of the operator T_0, we get that $L_1(\cdot)\in H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$. Similarly, we get that $L_2(\cdot)\in H^{3/2-\varepsilon}(\partial\Omega)$ and $T_0(H^{3/2-\varepsilon}(\partial\Omega))\subset\mathcal{R}_1$ for every $\varepsilon>0$.

We are now able to prove Theorem 5.20.

Proof of Theorem 5.20. Let $\sigma\in\mathcal{N}_0$ and we eventually reduce the size of the neighborhood later on. Our first goal consists in computing explicitly the coefficient associated to p.v.$(\frac{1}{\sigma})$ in $M_b'(u_{q_*})$. Using Proposition 5.21 and Remark 5.18, we have

$$\begin{aligned}M_b'(u_{q_*})(\sigma) &= \frac{\partial\phi_k'(u_{q_*})}{\partial\nu}(\sigma)\frac{\partial\xi_1}{\partial\nu}(\sigma)+\frac{\partial\phi_k}{\partial\nu}(\sigma)\frac{\partial\xi_{1,b}'(u_{q_*})}{\partial\nu}(\sigma)\\ &\quad+\frac{\partial\phi_1'(u_{q_*})}{\partial\nu}(\sigma)\frac{\partial\xi_k}{\partial\nu}(\sigma)+\frac{\partial\phi_1}{\partial\nu}(\sigma)\frac{\partial\xi_{k,b}'(u_{q_*})}{\partial\nu}(\sigma)\\ &= -2\left(\frac{\partial\phi_k}{\partial\nu}(0)a_1\text{p.v.}\left(\frac{1}{\sigma}\right)+\mathcal{P}_0(\sigma)\right)\left(\frac{\partial\xi_1}{\partial\nu}(0)+O(\sigma)\right)\\ &\quad-2\left(\frac{\partial\xi_1}{\partial\nu}(0)a_1\text{p.v.}\left(\frac{1}{\sigma}\right)+\mathcal{P}_0(\sigma)\right)\left(\frac{\partial\phi_k}{\partial\nu}(0)+O(\sigma)\right)\\ &\quad-2\left(\frac{\partial\phi_1}{\partial\nu}(0)a_1\text{p.v.}\left(\frac{1}{\sigma}\right)+\mathcal{P}_0(\sigma)\right)\left(\frac{\partial\xi_k}{\partial\nu}(0)+O(\sigma)\right)\\ &\quad-2\left(\frac{\partial\xi_k}{\partial\nu}(0)a_1\text{p.v.}\left(\frac{1}{\sigma}\right)+\mathcal{P}_0(\sigma)\right)\left(\frac{\partial\phi_1}{\partial\nu}(0)+O(\sigma)\right)\\ &= -4a_1\left(\frac{\partial\phi_k}{\partial\nu}(0)\frac{\partial\xi_1}{\partial\nu}(0)+\frac{\partial\phi_1}{\partial\nu}(0)\frac{\partial\xi_k}{\partial\nu}(0)\right)\text{p.v.}\left(\frac{1}{\sigma}\right)+\mathcal{P}_0(\sigma).\end{aligned}$$

where $\mathcal{P}_0(\sigma)$ denotes any function belonging to $H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$ in some open neighborhood of zero. Then, we have

$$M_b'(u_{q_*})(\sigma)=-4a_1M(0)\text{p.v}\left(\frac{1}{\sigma}\right)+\mathcal{P}_0(\sigma). \tag{5.68}$$

Since $M\equiv 0$ on $\partial\Omega$, we have in particular $M(0)=0$. In consequence, there is not any term in p.v$(\frac{1}{\sigma})$ in $M_b'(u_{q_*})$.

The next step consists in identifying the least regular term of $\mathcal{P}_0$. We begin by writing that

$$
\begin{aligned}
& M_b'(u_{q_*})(\sigma) \\
= \; & -\left(2a_1\frac{\partial}{\partial\tau}\left(\frac{\partial\phi_k}{\partial\nu}\right)(0)\ln|\sigma| + 2\left\{a_1\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial\phi_k}{\partial\nu}\right) + (a_1+2a_2)\lambda_k\frac{\partial\phi_k}{\partial\nu}\right\}(0)\sigma\ln|\sigma|\right. \\
& \left. - a_1\frac{\partial\phi_k}{\partial\nu}(0)L_1(\sigma) - a_1\frac{\partial}{\partial\tau}\left(\frac{\partial\phi_k}{\partial\nu}\right)(0)L_2(\sigma) + \mathcal{R}_1(\sigma)\right) \\
& \times\left(\frac{\partial\xi_1}{\partial\nu}(0) + \frac{\partial}{\partial\tau}\left(\frac{\partial\xi_1}{\partial\nu}\right)(0)\sigma + O(\sigma^2)\right) \\
& -\left(2a_1\frac{\partial}{\partial\tau}\left(\frac{\partial\xi_1}{\partial\nu}\right)(0)\ln|\sigma| + 2\left\{a_1\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial\xi_1}{\partial\nu}\right) + (a_1+2a_2)\lambda_k\frac{\partial\xi_1}{\partial\nu})\right\}(0)\sigma\ln|\sigma|\right. \\
& \left. - a_1\frac{\partial\xi_1}{\partial\nu}(0)L_1(\sigma) - a_1\frac{\partial}{\partial\tau}\left(\frac{\partial\xi_1}{\partial\nu}\right)(0)L_2(\sigma) + \mathcal{R}_1(\sigma)\right) \\
& \times\left(\frac{\partial\phi_k}{\partial\nu}(0) + \frac{\partial}{\partial\tau}\left(\frac{\partial\phi_k}{\partial\nu}\right)(0)\sigma + O(\sigma^2)\right) \\
& -\left(2a_1\frac{\partial}{\partial\tau}\left(\frac{\partial\phi_1}{\partial\nu}\right)(0)\ln|\sigma| + 2\left\{a_1\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial\phi_1}{\partial\nu}\right) + (a_1+2a_2)\lambda_1\frac{\partial\phi_1}{\partial\nu}\right\}(0)\sigma\ln|\sigma|\right. \\
& \left. - a_1\frac{\partial\phi_1}{\partial\nu}(0)L_1(\sigma) - a_1\frac{\partial}{\partial\tau}\left(\frac{\partial\phi_1}{\partial\nu}\right)(0)L_2(\sigma) + \mathcal{R}_1(\sigma)\right) \\
& \times\left(\frac{\partial\xi_k}{\partial\nu}(0) + \frac{\partial}{\partial\tau}\left(\frac{\partial\xi_k}{\partial\nu}\right)(0)\sigma + O(\sigma^2)\right) \\
& -\left(2a_1\frac{\partial}{\partial\tau}\left(\frac{\partial\xi_k}{\partial\nu}\right)(0)\ln|\sigma| + 2\left\{a_1\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial\xi_k}{\partial\nu}\right) + (a_1+2a_2)\lambda_1\frac{\partial\xi_k}{\partial\nu}\right\}(0)\sigma\ln|\sigma|\right. \\
& \left. - a_1\frac{\partial\xi_k}{\partial\nu}(0)L_1(\sigma) - a_1\frac{\partial}{\partial\tau}\left(\frac{\partial\xi_k}{\partial\nu}\right)(0)L_2(\sigma) + \mathcal{R}_1(\sigma)\right) \\
& \times\left(\frac{\partial\phi_1}{\partial\nu}(0) + \frac{\partial}{\partial\tau}\left(\frac{\partial\phi_1}{\partial\nu}\right)(0)\sigma + O(\sigma^2)\right).
\end{aligned}
$$

Rearranging the terms and using Remark 5.18, we get

$$
\begin{aligned}
& M_b'(u_{q_*})(\sigma) \\
= \; & -a_1\Big(\frac{\partial M}{\partial\tau}(0)(2\ln|\sigma| + \sigma L_1(\sigma) + L_2(\sigma)) + M(0)L_1(\sigma) + 2\frac{\partial^2 M}{\partial\tau^2}(0)\sigma\ln|\sigma|\Big) \\
& -(4a_1+8a_2)\left\{\lambda_1\frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\xi_k(0)}{\partial\nu} + \lambda_k\frac{\partial\phi_k(0)}{\partial\nu}\frac{\partial\xi_1(0)}{\partial\nu}\right\}\sigma\ln|\sigma| + \mathcal{R}_1(\sigma).
\end{aligned}
$$

Since $M(0)\equiv 0$ on $\partial\Omega$, we have $\frac{\partial M}{\partial\tau}(0)\equiv 0$ and $\frac{\partial^2 M}{\partial\tau^2}(0)=0$ on $\partial\Omega$. As a consequence, the above equation reduces equation (5.61).

□

5.5.4.2 Contribution of $M_d'(u_{q_*})$

We prove in this section the following theorem regarding the Taylor expansion of $M_d'(u_{q_*})$ in an open neighborhood of zero.

Theorem 5.22. *There exists an open neighborhood of zero $\mathcal{N}_2\subset\mathcal{N}_0$ such that, if $\sigma\in\mathcal{N}_2$, one has*

$$
M_d'(u_{q_*})(\sigma) = \frac{1}{\pi}\mu(q_*)\frac{\partial\phi_1}{\partial\nu}(0)\frac{\partial\phi_k}{\partial\nu}(0)\sigma\ln|\sigma| + \mathcal{R}_1. \tag{5.69}
$$

The proof of this theorem is based on the following proposition.

Proposition 5.23. *We keep the notations above, then we have*

$$\frac{\partial \xi'_{1,d}(u_{q_*})}{\partial \nu}(\sigma) = -(2a_1+4a_2)\mu(q_*)\frac{\partial \phi_1}{\partial \nu}(0)\sigma \ln|\sigma| + \mathcal{R}_1, \tag{5.70}$$

$$\frac{\partial \xi'_{k,d}(u_{q_*})}{\partial \nu}(\sigma) = -(2a_1+4a_2)\mu(q_*)\frac{\partial \phi_k}{\partial \nu}(0)\sigma \ln|\sigma| + \mathcal{R}_1. \tag{5.71}$$

Proof of Theorem 5.22. We note that $2a_1+4a_2 = -\frac{1}{2\pi}$. Assuming Proposition 5.23. We have

$$\begin{aligned} M'_d(u_{q_*})(\sigma) &= \frac{\partial \phi_k}{\partial \nu}\frac{\partial \xi'_{1,d}(u_{q_*})}{\partial \nu} + \frac{\partial \phi_1}{\partial \nu}\frac{\partial \xi'_{k,d}(u_{q_*})}{\partial \nu} \\ &= \left(\frac{\partial \phi_k}{\partial \nu}(0)+O(\sigma)\right)\left(\frac{1}{2\pi}\mu(q_*)\frac{\partial \phi_1}{\partial \nu}(0)\sigma\ln|\sigma|+\mathcal{R}_1\right) \\ &\quad + \left(\frac{\partial \phi_1}{\partial \nu}(0)+O(\sigma)\right)\left(\frac{1}{2\pi}\mu(q_*)\frac{\partial \phi_k}{\partial \nu}(0)\sigma\ln|\sigma|+\mathcal{R}_1\right) \\ &= \frac{1}{\pi}\mu(q_*)\frac{\partial \phi_1}{\partial \nu}(0)\frac{\partial \phi_k}{\partial \nu}(0)\sigma\ln|\sigma|+\mathcal{R}_1. \end{aligned}$$

□

Recall now the system verified by $\xi'_{1,d}$.

$$\begin{cases} -(\Delta+\lambda_k)\xi'_{1,d}(u_{q_*}) = \lambda'_k(u_{q_*})\xi_1 + \mu(q+q_*)\phi'_1(u_{q_*}) & \text{in } \Omega, \\ \xi'_{1,d}(u_{q_*}) = 0 & \text{on } \partial\Omega, \\ \int_\Omega \phi_k \xi'_{1,d}(u_{q_*}) = 0. \end{cases} \tag{5.72}$$

The function $\xi'_{k,d}$ verifies a similar system by exchanging the indices 1 and k and we will omit the corresponding argument.

Remark 5.19. By classical elliptic regularity theory presented in [67], we know that $\phi'_1 \in H^{1-\epsilon}(\Omega)$, and then $\xi'_{1,d} \in H^{3-\epsilon}(\Omega)$. By taking the trace, we have $\frac{\partial \xi'_{1,d}}{\partial \nu} \in H^{3/2-\epsilon}(\partial\Omega)$. A straightforward computation shows that the last term $\sigma\ln|\sigma|$ in our expansion of $M'_b(u_{q^*})$ is in $H^{3/2-\epsilon}(\partial\Omega)$. Hence, it is necessary to compute exactly the first singular term of $\frac{\partial \xi'_{1,d}}{\partial \nu}$.

Remark 5.20. The term $M'_d(u_{q^*})$ cannot be treated by a direct functional analysis argument : if there were a family of functional spaces X^s with a well established theory of elliptic equations such as that for Sobolev spaces and if there would exist $s_1 \neq s_2$ such that $\mathcal{H}_0 \in X^{s_1}$ and $\ln|\sigma| \in X^{s_2}$, then $\frac{\partial \xi'_{1,d}}{\partial \nu}$ and $\sigma\ln|\sigma|$ would not be in the same X^s. However, we cannot distinguish these two functions even in the family of Besov spaces. We can also note that $\mathcal{H}_0$ is a bounded variation function and $\ln|\sigma|$ is not, but elliptic theory in the space of bounded variation functions is not easy. For these reasons, it seems that an exact computation of the first term of $\frac{\partial \xi'_{1,d}}{\partial \nu}$ is necessary.

Let us first prove the following technical lemma, which expresses integrals over Ω by means of boundary integrals over $\partial\Omega$.

Lemma 5.24. *Let k, m be two distinct positive integers. Assume that a function h verifies $(\Delta+\lambda_k)h = 0$ in Ω. Then we have*

$$\frac{\partial}{\partial \nu_p}\int_\Omega h(q)G_m(p,q)dq = \frac{-1}{\lambda_m-\lambda_k}\left((E_m-E_k)(h|_{\partial\Omega}) + (K_k^*-K_m^*)\left(\frac{\partial h}{\partial \nu}\right)\right), \tag{5.73}$$

where $G_m(\cdot,\cdot)$ is the fundamental solution of the Helmhotz equation corresponding to λ_m and verifying the Sommerfeld condition and the operators E_k, E_m, K_k^*, K_m^* are defined in Subsection 5.8.1.

Proof of Lemma 5.24. Green's second formula says that

$$\int_\Omega (g_1\Delta(g_2) - g_2\Delta(g_1)) = \int_{\partial\Omega} \left(g_1 \frac{\partial g_2}{\partial \nu} - g_2 \frac{\partial g_1}{\partial \nu} \right) d\sigma(q),$$

where g_1, g_2 are arbitrary functions such that the above integrals exist. Choose $g_1 = h$ and $g_2 = cG_k$ where c is a real number to be determined later. We have

$$c\int_\Omega h(q)[(\Delta+\lambda_k)G_m(p,q)]dq = c\int_{\partial\Omega} \left(h(q)\frac{\partial G_m}{\partial \nu_q}(p,q) - G_m(p,q)\frac{\partial h}{\partial \nu_q} \right) d\sigma(q). \tag{5.74}$$

Since $(\Delta+\lambda_m)G_m(p,q) = \delta_p$, we then get

$$\begin{aligned} & ch(p) + c(\lambda_k - \lambda_m)\int_\Omega h(q)G_m(p,q)dq \\ = \;& c\int_{\partial\Omega} \left(h(q)\frac{\partial G_m}{\partial \nu_q}(p,q) - G_m(p,q)\frac{\partial h}{\partial \nu_q} \right) d\sigma(q). \end{aligned}$$

Setting $c := \frac{1}{\lambda_m - \lambda_k}$, we then get, for $p \in \Omega$,

$$\int_\Omega h(q)G_m(p,q)dq = \frac{1}{\lambda_m - \lambda_k}\left(-D_m(h)(p) + S_m\left(\frac{\partial h}{\partial \nu}\right)(p) + h(p) \right), \tag{5.75}$$

where S_m, D_m are respectively the single-layer and double-layer potentials associated to G_m (cf. Subsection 5.8.1).

By applying the normal derivative operator to the two side terms of equation (5.75) and then taking into account the jump relations (5.102) and $\left(\frac{1}{2}I + K_k^*\right)\frac{\partial}{\partial\nu}h = E_k(h)$, we get that

$$\begin{aligned} & \frac{\partial}{\partial\nu}\int_\Omega h(q)G_m(p,q)dq \\ = \;& \frac{1}{\lambda_m-\lambda_k}\frac{\partial}{\partial\nu}\left(-D_m(h)(p) + S_m\left(\frac{\partial h}{\partial\nu}\right)(p) + h(p) \right) \\ = \;& \frac{1}{\lambda_m-\lambda_k}\left(-E_m(h|_{\partial\Omega}) + \left(\frac{-1}{2} + K_m^*\right)\left(\frac{\partial h}{\partial\nu}\right) + \frac{\partial h}{\partial\nu} \right)(p) \\ = \;& \frac{1}{\lambda_m-\lambda_k}\left(-E_m(h|_{\partial\Omega}) + K_m^*\left(\frac{\partial h}{\partial\nu}\right) + \frac{1}{2}\frac{\partial h}{\partial\nu} \right)(p) \\ = \;& \frac{1}{\lambda_m-\lambda_k}\left(-E_m(h|_{\partial\Omega}) + K_m^*\left(\frac{\partial h}{\partial\nu}\right) + \left[E_k(h|_{\partial\Omega}) - K_k^*\frac{\partial h}{\partial\nu}\right] \right)(p) \\ = \;& \frac{1}{\lambda_m-\lambda_k}\left((E_k - E_m)(h|_{\partial\Omega}) + (K_m^* - K_k^*)\left(\frac{\partial h}{\partial\nu}\right) \right)(p). \end{aligned}$$

□

We are now able to provide an argument for Proposition 5.23.

Proof of Proposition 5.23. According to (5.106), we first easily get that the contribution of $\lambda_k'(u_{q_*})\xi_1$ to $\frac{\partial\xi_{1,d}'}{\partial\nu}(q_*)$ is a term of class C^2 and that

$$\left(\frac{1}{2}I + K_k^*\right)\frac{\partial\xi_{1,d}'}{\partial\nu}(p) = \frac{\partial}{\partial\nu_p}\int_\Omega \mu(q+q_*)\phi_1'(q)G_k(p,q)dq + \mathcal{R}_1. \tag{5.76}$$

We need the Taylor expansion of the right-hand side of (5.76) when a boundary point p belongs to an open neighborhood (in $\partial\Omega$) of q_* (i.e. $(0,0)$). For that purpose, we perform the following decomposition.

$$\begin{aligned}\frac{\partial}{\partial\nu_p}\int_\Omega \mu(q+q_*)\phi_1'(q)G_k(p,q)dq &= \int_\Omega \mu(q+q_*)\phi_1'(q)\frac{\partial G_k}{\partial\nu_p}(p,q)dq\\ &= \mu(p+q_*)I_1(p)+I_2(p),\end{aligned}$$

where

$$I_1(p)=\int_\Omega \phi_1'(q)\frac{\partial G_k}{\partial\nu_p}(p,q)dq=\frac{\partial}{\partial\nu_p}\int_\Omega \phi_1'(q)G_k(p,q)dq,$$

and

$$I_2=\int_\Omega [\mu(q+q_*)-\mu(p+q_*)]\phi_1'(q)\frac{\partial G_k}{\partial\nu_p}(p,q)dq.$$

We first treat $I_1(p)$. Since ϕ_1' verifies $(\Delta+\lambda_1)\phi_1'=0$, we can apply Lemma 5.24 and we get

$$I_1(p)=\frac{-1}{\lambda_k-\lambda_1}\left((E_k-E_1)\left(\phi_1'|_{\partial\Omega}\right)+(K_1^*-K_k^*)\left(\frac{\partial\phi_1'}{\partial\nu}\right)\right). \tag{5.77}$$

Using the arc-length σ, recall that $p=O(\sigma)$. Thus, we have $p+q_*=q_*+O(\sigma)$ and we write $I_1(\sigma)$ for $I_1(p(\sigma))$. According to (5.115), we first deduce that

$$\begin{aligned}\frac{1}{\lambda_k-\lambda_1}(E_k-E_1)(\phi_1') &= \frac{1}{\lambda_k-\lambda_1}(a_1+2a_2)(\lambda_k-\lambda_1)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma|+\mathcal{R}_2\\ &= (a_1+2a_2)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma|+\mathcal{R}_2.\end{aligned} \tag{5.78}$$

As K_k^* and K_1^* have the same principal part, by Lemma 5.32, we know that $(K_k^*-K_1^*)\frac{\partial\phi_1'}{\partial\nu}$ is a 2-regular term. Then,

$$I_1(\sigma)=-(a_1+2a_2)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma|+\mathcal{R}_2. \tag{5.79}$$

We now treat $I_2(p)$. Taking into account the Taylor expansion of μ at $p+q*$, we can rewrite $I_2(p)=d\mu(p+q_*)J_2(p)+R(p)$ where

$$J_2(p)=\int_\Omega (q-p)\frac{\partial G_k}{\partial\nu_p}(p,q)\phi_1'(q)dq,$$

and

$$R(p)=\int_\Omega O(\|q-p\|^2)\frac{\partial G_k}{\partial\nu_p}(p,q)\phi_1'(q)dq.$$

Since $R(\cdot)$ is a more regular term than $J_2(\cdot)$, it is enough to prove that J_2 is of class C^1.

Note that $J_2=\int_\Omega H(p,q)\phi'(q)dq$ with $H(\cdot,\cdot)$ the convolution kernel given by $H(p,q):=(q-p)\frac{\partial G_k}{\partial\nu_p}(p,q)$, $p\neq q$. The kernel H is no longer singular (it is actually uniformly bounded on its domain of definition) and straightforward computations yield that H defines a pseudodifferential operator of class $-3/2$. Recall that $\mathcal{H}_0\in H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$, we deduce that $\phi_1'\in H^{1-\varepsilon}(\Omega)$ for every $\varepsilon>0$, then $I_2\in H^{5/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$. Then $\sigma\mapsto J_2(p(\sigma))$ admits a continuous derivative in an open neighborhood of zero. We conclude that the contribution of J_2 to $\frac{\partial\xi_{1,d}'}{\partial\nu}(\sigma)$ yields an $\mathcal{R}_1$ term.

By Theorem 5.31, we finally get

$$\frac{\partial\xi_{1,d}'}{\partial\nu}(\sigma)=-(2a_1+4a_2)\mu(q_*)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma|+\mathcal{R}_1. \tag{5.80}$$

□

5.5.5 Proof of Proposition 5.17

Collecting the results of Theorems 5.20 and 5.22 in (5.43), we get that, for σ in some open neighborhood of zero, one has

$$\begin{aligned} M'(u_{q_*})(\sigma) &= \frac{1}{\pi}\left\{\lambda_1 \frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\xi_k(0)}{\partial\nu} + \lambda_k \frac{\partial\phi_k(0)}{\partial\nu}\frac{\partial\xi_1(0)}{\partial\nu}\right. \\ &\quad \left. +\mu(q_*)\frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\phi_k(0)}{\partial\nu}\right\}\sigma\ln|\sigma| + \mathcal{R}_1 \\ &= -\mathcal{H}_0\frac{\partial M}{\partial\nu}(0). \end{aligned} \tag{5.81}$$

The left-hand side of (5.81) is continuous at $\sigma = 0$, which implies that $\frac{\partial M}{\partial\nu}(0) = 0$. Then the left-hand side of (5.81) must be of class C^1 at $\sigma = 0$, implying that the coefficient of $\sigma\ln|\sigma|$ must also be equal to zero. We finally get that

$$\lambda_1 \frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\xi_k(0)}{\partial\nu} + \lambda_k \frac{\partial\phi_k(0)}{\partial\nu}\frac{\partial\xi_1(0)}{\partial\nu} + \mu(q_*)\frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\phi_k(0)}{\partial\nu} = 0, \tag{5.82}$$

and, since q_* is an arbitrary point of $\partial\Omega$, we get

$$\lambda_1 \frac{\partial\phi_1}{\partial\nu}(q)\frac{\partial\xi_k}{\partial\nu}(q) + \lambda_k \frac{\partial\phi_k}{\partial\nu}(q)\frac{\partial\xi_1}{\partial\nu}(q) + \mu(q)\frac{\partial\phi_1}{\partial\nu}(q)\frac{\partial\phi_k}{\partial\nu}(q) = 0 \quad \text{on } \partial\Omega. \tag{5.83}$$

Consider now equations (5.38) and (5.83) as a linear system with $\frac{\partial\xi_1}{\partial\nu}(q)$ and $\frac{\partial\xi_k}{\partial\nu}(q)$ as unknowns. After an elementary algebraic manipulation, we have, for every $q \in \partial\Omega$,

$$\frac{\partial\phi_k}{\partial\nu}(q)\left\{\frac{\partial\xi_1}{\partial\nu}(q) - \frac{1}{\lambda_1-\lambda_k}\mu(q)\frac{\partial\phi_1}{\partial\nu}(q)\right\} = 0, \tag{5.84}$$

$$\frac{\partial\phi_1}{\partial\nu}(q)\left\{\frac{\partial\xi_k}{\partial\nu} - \frac{1}{\lambda_k-\lambda_1}\mu(q)\frac{\partial\phi_k}{\partial\nu}(q)\right\} = 0 \quad . \tag{5.85}$$

As ϕ_1 and ϕ_k are eigenfunctions of $-\Delta_\Omega^D$, by Holmgren uniqueness theorem (see [100, Proposition 4.3, page 433]), their normal derivatives cannot be equal to zero on a subset of $\partial\Omega$ with non null measure. Then, by a simple density argument, we get

$$\frac{\partial\xi_1}{\partial\nu}(q) - \frac{1}{\lambda_1-\lambda_k}\mu(q)\frac{\partial\phi_1}{\partial\nu}(q) = 0 \qquad \text{on } \partial\Omega, \tag{5.86}$$

$$\frac{\partial\xi_k}{\partial\nu}(q) - \frac{1}{\lambda_k-\lambda_1}\mu(q)\frac{\partial\phi_k}{\partial\nu}(q) = 0 \qquad \text{on } \partial\Omega. \tag{5.87}$$

What we have proved so far is that, if Property (B_k), $k > 1$, is not generic then a certain property (C_k) is not as well, where the latter property is defined exactly as in Definition 5.11 except that the function M defined in (5.30) is replaced by the function $S : \partial\Omega \to \mathbb{R}^2$ defined by

$$S(q) := \frac{\partial\xi_1}{\partial\nu}(q) - \frac{1}{\lambda_1-\lambda_k}\mu(q)\frac{\partial\phi_1}{\partial\nu}(q) \text{ for } q \in \partial\Omega. \tag{5.88}$$

As in Proposition 5.17, it now amounts to prove that the function S defined in (5.88) cannot be identically equal to zero on any $E_\rho(\Omega)$ with $\rho > 0$. We can follow the same strategy developed in Subsection 5.5.2 and use the computations made in Section 5.5.4.

Reasoning by contradiction, we assume that $S \equiv 0$ on $\partial\Omega$. Taking the shape differentiation of that equation and using a variation u_{q_*} for an arbitrary $q_* \in \partial\Omega$, we get

$$\begin{aligned} & \frac{\partial \xi_1'(u_{q_*})}{\partial \nu}(q) - \frac{1}{\lambda_1 - \lambda_k}\mu(q)\frac{\partial \phi_1'(u_{q_*})}{\partial \nu}(q) \\ = \; & -\mathcal{H}_0 \frac{\partial}{\partial \nu}\left\{ \frac{\partial \xi_1}{\partial \nu}(q) - \frac{1}{\lambda_1 - \lambda_k}\mu(q)\frac{\partial \phi_1}{\partial \nu}(q) \right\}. \end{aligned} \tag{5.89}$$

Using Propositions 5.21 and 5.23, we have, at $\sigma = 0$,

$$\begin{aligned} & -2\left(\frac{\partial \xi_1}{\partial \nu}(0) - \frac{1}{\lambda_1 - \lambda_k}\frac{\partial \phi_1}{\partial \nu}(0)\mu(q_*) \right) a_1 \text{p.v.}\left(\frac{1}{\sigma}\right) \\ & +2\left(\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) - \frac{1}{\lambda_1 - \lambda_k}\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\mu(q_*) \right) a_1 \ln|\sigma| + O(\sigma \ln|\sigma|) \\ = \; & -\mathcal{H}_0 \frac{\partial}{\partial \nu}\left\{ \frac{\partial \xi_1}{\partial \nu} - \frac{1}{\lambda_1 - \lambda_k}\frac{\partial \phi_1}{\partial \nu}\mu(q) \right\}(0). \end{aligned}$$

By (5.86), we simplify the previous equation and get

$$\begin{aligned} & 2\left(\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) - \frac{1}{\lambda_1 - \lambda_k}\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\mu(q_*) \right) a_1 \ln|\sigma| + O(\sigma \ln|\sigma|) \\ = \; & \mathcal{H}_0 \frac{\partial}{\partial \nu}\left\{ \frac{\partial \xi_1}{\partial \nu} - \frac{1}{\lambda_1 - \lambda_k}\frac{\partial \phi_1}{\partial \nu}\mu(q(\sigma)) \right\} \end{aligned}$$

Since the right-hand side remains bounded in in neighborhood of $\sigma = 0$, it is necessary that

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) - \frac{1}{\lambda_1 - \lambda_k}\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\mu(q_*) = 0. \tag{5.90}$$

On the other hand, by taking the tangent derivative of (5.86) at $q = q_*$, we have

$$\begin{aligned} \frac{\partial S}{\partial \tau}(q(\tau)) &= \frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) \\ & \quad - \frac{1}{\lambda_1 - \lambda_k}\left(\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\mu(q_*) + \frac{\partial \phi_1}{\partial \nu}(0) d\mu(q_*)\cdot \tau_0 \right) \\ &= 0, \end{aligned} \tag{5.91}$$

where τ_0 is the unit tangent vector on $\partial\Omega$ at the point q_*. From (5.90) and (5.91), we end up with

$$\frac{\partial \phi_1}{\partial \nu}(0) d\mu(q_*)\cdot \tau_0 = 0. \tag{5.92}$$

As the previous reasoning is valid almost everywhere on $\partial\Omega$, we have

$$\frac{\partial \phi_1}{\partial \nu}(q) d\mu(q)\cdot \tau_q = 0, \quad \text{for } q \in \partial\Omega. \tag{5.93}$$

By condition (5.36) and by continuity of the map $q \mapsto d\mu(q)\cdot\tau_q$ for $q \in \partial\Omega$, we get that $\frac{\partial \phi_1}{\partial \nu}$ is equal to zero on an open neighborhood of $\bar{q}$ on $\partial\Omega$ (defined in (5.36)). This is not possible by Holmgren uniqueness theorem. We finally derived a contradiction and Proposition 5.17 is now proved.

5.6 Conclusion, conjectures, perspectives

We recapitulate all the controllability results known for (5.3) in the following arrays.

	Spectral controllability in time T of (5.3)
$1D$ $\Omega = (0,1)$	*yes* under **(H1)** $\forall T > 0$ *no* under **no(H1)**
$2D$	*yes* under **(H3)** (i.e. generically with respect to (Ω, μ)) with $T > T_{min}(\Omega)$ *no* under **(H3)** and **(H4)** with $T < T_{min}(\Omega)$ *no* under **no (H3)**
$3D$	*no* under **(H4)**

	Exact controllability in time T of (5.3)
$1D$ $\Omega = (0,1)$	*yes* under **(H2)** (i.e. generically with respect to μ) in $H^3_{(0)}((0,1),\mathbb{C})$ with $L^2((0,T),\mathbb{R})$-controls , $\forall T > 0$ *no* under **no(H1)**
$2D$	*yes* under **(H3)** and **(H4)** in abstract spaces with$T > T_{min}(\Omega)$ *no* under **no (H3)**
$3D$	*no* under **(H4)**

In these arrays, we have used the notation

$$T_{min}(\Omega) := 2\pi d(\Omega)$$

where $d(\Omega) > 0$ is such that

$$\sharp\{k \in \mathbb{N}^*; \lambda_k - \lambda_1 \in [0,t]\} \sim d(\Omega)t, \text{ when } t \to +\infty.$$

and the assumptions

(H1) : $\langle \mu\varphi_1, \varphi_k\rangle_{L^2(\Omega)} \neq 0$, for every $k \in \mathbb{N}^*$,

(H2) : there exists $c_1, c_2 > 0$ such that,

$$\frac{c_1}{k^3} \leq |\langle \mu\varphi_1, \varphi_k\rangle| \leq \frac{c_2}{k^3}, \forall k \in \mathbb{N}^*,$$

(H3) : any eigenvalue λ of $-\Delta^D_\Omega$ has multiplicity $m \leq n$ ($n = 2, 3$ is the space dimension : $\Omega \subset \mathbb{R}^n$) and the vectors $\langle \mu\phi_1, \phi_{k_1}\rangle$,...,$\langle \mu\phi_1, \phi_{k_m}\rangle$ are linearly independant in $\mathbb{R}^n$, where $k_1 < ... < k_m$ and $\phi_{k_1}, ..., \phi_{k_m}$ are the eigenvectors associated to λ.

(H4) : there exists $\tilde{\mu} \in C^0(\overline{\Omega}, \mathbb{R})$ such that $\mu(q) = \tilde{\mu}(q)e_1$.

The assumption **(H4)** is not necessary for the non spectral controllability of (5.3) in small time in 2D (see Remark 5.7). We conjecture that, in 2D, the system (5.3) is not spectral controllable in small time under **(H3)**. This is an open problem.

Similarly, the assumption **(H4)** is not necessary for the non spectral controllability of (5.3) in any time $T > 0$ in 3D. We conjecture that, in 3D, and with any time $T > 0$, the system (5.3) is not spectral controllable in time T under **(H3)**. This is an open problem.

A strategy to prove these conjectures could be the adaptation of Haraux and Jaffard's result (Theorem 5.10) to vector exponential families.

5.7 Appendix : Shape differentiation

The material presented here is borrowed from [89] and [80].

5.7.1 Main definitions

Let Ω be a domain in $\mathbb{D}_3$. For a positive integer l, we consider perturbations u in the space $W^{l,\infty}(\Omega, \mathbb{R}^2)$ with norm

$$\|u\|_{l,\infty} := \text{supess}\{|D^\alpha u(x)|;\ 0 \leq \alpha \leq l,\ x \in \Omega\}.$$

Then, the domain $\Omega + u$ is defined by

$$\Omega + u := (\text{Id} + u)(\Omega) = \{x + u(x),\ x \in \Omega\}.$$

Lemma 5.25 (cf. [89]). *Let $l \in \mathbb{N}^*$ and $u \in W^{l,\infty}(\Omega, \mathbb{R}^2)$ be such that $\|u\|_{l,\infty} \leq 1/2$. Then, the map $Id + u$ is invertible. Furthermore, there exits $w \in W^{l,\infty}(\Omega + u, \mathbb{R}^2)$ such that $(Id + u)^{-1} = Id + w$ and $\|w\|_{l,\infty} \leq C_l \|u\|_{l,\infty}$ where C_l is a constant independent on u.*

Remark 5.21. According to this result, if Ω is of class C^j, we can choose $l = j + 1$ so that the new domain $\Omega + u$ is also of class C^j. In particular, if we need domains of class C^3, we can choose $W^{4,\infty}(\Omega, \mathbb{R}^2)$ as the perturbation space.

We now consider a function

$$v :\ u \in W^{l,\infty}(\Omega, \mathbb{R}^2) \to\ v(u) \in W^{m,r}(\Omega + u)$$

where $1 \leq r < \infty$ and $m \leq l$ are integers. In practice, $v(u)$ is solution of a suitable problem, which depends on the perturbation function u. We are interested in the study of the regularity of the function $v(u)$ with respect to the perturbation function u.

Définition 5.14 (First order local variation). Let $k \geq m \geq 1$ and $1 \leq r < \infty$. We say that the function $v(u)$ has a first order local variation at $u = 0$ on $W^{m,\,r}(\Omega+u)$ for all $u \in W^{l,\infty}(\Omega, \mathbb{R}^2)$ if there exists a linear map $v'(u)$ from $u \in W^{l,\infty}(\Omega, \mathbb{R}^2)$ to $v'(u) \in W^{m-1,\,r}_{\text{loc}}(\Omega)$ such that, for every open set $\omega \subset\subset \Omega$,

$$v(u) = v(0) + v'(u) + \theta(u) \quad \text{in } \omega,$$

when $\|u\|_{l,\infty}$ is small enough and

$$\frac{\|\theta(u)\|_{m-1,r}}{\|u\|_{l,\infty}} \to 0 \quad \text{as } \|u\|_{l,\infty} \to 0.$$

Remark 5.22. The first order local variation can also be defined as

$$v'(u) = \lim_{t\to 0} \frac{v(tu)|_\omega - v(0)|_\omega}{t} \quad \text{in } W^{m-1,r}(\omega),$$

where $\omega \subset\subset \Omega$.

The following theorem provides sufficient conditions for the existence of the first order local variation.

Theorem 5.26 (cf. [89]). *Let Ω be a $C^{0,1}$ domain. Consider the map $u \to v(u) \in W^{m,r}(\Omega+u)$ defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega,\mathbb{R}^2)$, with $k \geq m \geq 1$ and $1 \leq r < \infty$. Assume that there exists a linear continuous map $u \in W^{k,\infty}(\Omega) \to \dot{v}(u) \in W^{m,\,r}(\Omega)$ such that*

$$v(u) \circ (Id + u) = v(0) + \dot{v}(u) + \theta(u) \quad \text{in } W^{m,\,r}(\Omega),$$

for all $u \in W^{k,\infty}(\Omega,\mathbb{R}^2)$ small enough, where

$$\frac{\|\theta(u)\|_{m-1,r}}{\|u\|_{k,\infty}} \to 0 \quad \text{as } \|u\|_{k,\infty} \to 0.$$

Furthermore, we assume that for every $u \in W^{k,\infty}(\Omega,\mathbb{R}^2)$ small enough,

$$v(u) = 0 \quad \text{on } \partial\Omega + u.$$

Then, for each $\omega \subset\subset \Omega$, the function $u \to v(u)|_\omega \in W^{m-1,r}(\omega)$ defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega,\mathbb{R}^2)$ is differentiable at $u = 0$.

Moreover, the map $u \to v(u)|_\omega$ has a first order local variation and this variation at $u = 0$ in the direction u_1 denoted by $v'(u_1)$ verifies $v'(u_1) \in W^{m-1,r}(\Omega)$ and

$$v'(u_1) = - < u_1, \nu > \frac{\partial v(0)}{\partial \nu} \quad \text{on } \partial\Omega. \tag{5.94}$$

5.7.2 Regularity of the eigenvalues and eigenfunctions

By applying [81, Theorem 3] in the same way as in [80], we get the following result.

Theorem 5.27. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain of class C^1. Let λ be an eigenvalue of multiplicity h of $-\Delta_\Omega^D$, with associated orthonormal eigenfunctions $y_1, ..., y_h$. Then, there exist h real-valued continuous functions, $u \mapsto \lambda_i^{\Omega+u}$, and h continuous functions with values in $H^2 \cap H_0^1(\Omega,\mathbb{R})$, $u \mapsto y_i(u)$, for $i = 1, ..., h$, defined in a neighborhood U of $u = 0$ in $W^{4,\infty}(\Omega,\mathbb{R}^3)$ such that the following properties hold,*

- *$\lambda_i^\Omega = \lambda$ for $i = 1, ..., h$,*
- *for every $u \in U$, $\varphi_i^{\Omega+u} := y_i(u) \circ (I + u)^{-1}$ is an eigenfunction of $-\Delta_{\Omega+u}^D$ associated to the eigenvalue $\lambda_i^{\Omega+u}$,*
- *for every $u \in U$, the family $(\varphi_1^{\Omega+u} ..., \varphi_h^{\Omega+u})$ is orthonormal in $L^2(\Omega + u,\mathbb{R})$,*
- *for each open interval $I \subset \mathbb{R}$, such that the intersection of I with the set of eigenvalues of $-\Delta_\Omega^D$ contains only λ, there exists a neighborhood $U_I \subset U$ such that, for every $u \in U_I$, there exist exactly h eigenvalues (counting the multiplicity), $\lambda_i^{\Omega+u}$, $1 \leq i \leq h$, of $-\Delta_{\Omega+u}^D$ contained in I,*
- *for each $u \in W^{2,\infty}(\Omega,\mathbb{C})$ and $1 \leq i \leq h$, the map*

$$\begin{array}{ccccc} \mathbb{R} & \to & \mathbb{R} & \times & H^2 \cap H_0^1(\Omega,\mathbb{R}) \\ t & \mapsto & (\lambda_i^{\Omega+tu} & , & y_i(tu)) \end{array}$$

is analytic in a neighborhood of $t = 0$.

5.7.3 Local variations of the eigenvalues and eigenfunctions

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^1. Let λ be an eigenvalue of multiplicity h of $-\Delta_\Omega^D$, with associated orthonormal eigenfunctions $y_1, ..., y_h$. Let $\varphi_i(u) \in H^2 \cap H_0^1(\Omega + u, \mathbb{R})$, $i = 1, ..., h$ be the eigenfunctions of $-\Delta_{\Omega+u}^D$ associated to the eigenvalues $\lambda_i(u)$, $i = 1, ..., h$, where $\lambda_i(0) = \lambda$ for $i = 1, ..., h$.

According to the result of the previous section, the functions $t \mapsto \lambda_i(tu)$ and $t \mapsto \varphi_i(tu)$ are analytic in a neighborhood of 0. Let us denote by

$$\lambda_i'(u_0) \text{ (resp. } \left.\frac{d\varphi_i}{du}\right]_{u_0} \text{)}$$

the value of the directional derivative of λ_i (resp. φ_i) at $u = 0$ in the direction u_0,

$$\lambda_i'(u_0) := \lim_{t\to 0} \frac{\lambda_i(tu_0) - \lambda_i(0)}{t}.$$

For $i = 1, ..., h$, $\frac{d\varphi_i}{du}]_{u_0} \in H^2(\Omega, \mathbb{R})$ and, for every open subset $\omega \subset\subset \Omega$,

$$\left.\frac{d\varphi_i}{du}\right]_{u_0} := \lim_{t\to 0} \frac{\varphi_i(tu_0)|_\omega - \varphi_i(0)|_\omega}{t}, \text{ in } H^2(\omega, \mathbb{R}^3).$$

We have, for every $t \in \mathbb{R}$,

$$\begin{cases} -\Delta\varphi_i(tu_0) = \lambda_i(tu_0)\varphi_i(tu_0) & \text{in } \Omega + tu_0, \\ \varphi_i(tu_0) = 0 & \text{on } \partial(\Omega + tu_0), \\ \int_{\Omega+tu_0} |\varphi_i(tu_0)(q)|^2 dq = 1. \end{cases}$$

Thus, using classical results on shape differentiation (see [89]), we get

$$\begin{cases} -(\Delta + \lambda_i) \left.\frac{d\varphi_i}{du}\right]_{u_0} = \lambda_i'(u_0)\varphi_i & \text{in } \Omega, \\ \left.\frac{d\varphi_i}{du}\right]_{u_0} = -u_0.\nabla\varphi_i & \text{on } \partial\Omega, \\ \int_\Omega \varphi_i(q) \left.\frac{d\varphi_i}{du}\right]_{u_0} (q)dq = 0. \end{cases} \tag{5.95}$$

Remark 5.23. We note that all results stated above can be easily extended for C^3 domains and variations $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$.

5.8 Appendix : Dirichlet to Neumann map for the Helmholtz equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a connected boundary $\partial\Omega$ of class C^3 and outward unit normal ν. For $k > 0$, we consider the following problem

$$\begin{cases} (\Delta + k^2)u = F, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{5.96}$$

The goal of this section is to study the Dirichlet-to-Neumann map associated to (5.96) when $-k^2$ is an eigenvalue of the interior Dirichlet problem. In subsection 5.8.1, we recall some useful background results (see for instance [5, 31, 78, 101]). In subsection 5.8.2, we study precisely the Dirichlet to Neumann map associated to (5.96).

5.8.1 Preliminary results on Helmholtz equation

A standard approach for studying the Helmholtz equations consists in the representation of the solution using the single and double layer potentials respectively defined by

$$S_k(f)(p) := \int_{\partial\Omega} G_k(p,q) f(q) d\sigma(q), \quad \forall p \in \mathbb{R}^2 \backslash \partial\Omega, \tag{5.97}$$

and

$$D_k(f)(p) := \int_{\partial\Omega} \frac{\partial G_k(p,q)}{\partial \nu_q} f(q) d\sigma(q), \quad \forall p \in \mathbb{R}^2 \backslash \partial\Omega, \tag{5.98}$$

where $G_k(.,.)$ is the fundamental solution of the Helmholtz equation that satisfies the Sommerfeld condition and $f \in L^2(\partial\Omega)$. Here the notation $\frac{\partial}{\partial \nu_q}$ stands for the outward unit normal to $\partial\Omega$ at the point q. Then the solution of (5.96) is given by the third Green formula,

$$u = -S_k\left(\frac{\partial u}{\partial \nu}\right) + D_k(f) + F * G_k, \tag{5.99}$$

where

$$F * G_k(p) := \int_\Omega F(q) G_k(p,q)\ dq, \quad \forall p \in \Omega.$$

For the reader's convenience, we recall the following useful standard result which highlights the difference between the fundamental solution G_0 of the Laplace equation and G_k (see [1] and [77]).

5.8.1.1 Fundamental solution

Proposition 5.28. *Let $k > 0$. The fundamental solution for the Helmholtz equation is*

$$G_k(p,q) = -\frac{i}{4} H_0^1(k \mid p - q \mid) \tag{5.100}$$

where H_0^1 denotes the Hankel function of the first kind of order 0. If

$$G_0(p,q) := \frac{1}{2\pi} \ln \mid p - q \mid$$

is the fundamental solution of the Laplace equation, then we have

$$G_k(p,q) = G_0(p,q) + g_k(p,q), \tag{5.101}$$

where $g_k = g_k^{(1)} + g_k^{(2)}$ with

$$g_k^{(1)}(p,q) := -\frac{1}{2\pi} \ln\left(\frac{k}{2}\right) + \frac{1}{2\pi} \ln\left(\frac{k \mid p - q \mid}{2}\right) \sum_{j=1}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{k \mid p - q \mid}{2}\right)^{2j},$$

and

$$g_k^{(2)}(p,q) := -\frac{i}{4} J_0(k \mid p - q \mid) + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j!)^2} \psi(j+1) \left(\frac{k|p-q|}{2}\right)^{2j},$$

with ψ, the digamma function and J_0, the Bessel function of first kind.

5.8.1.2 Jump relations

Now, let us state jump relations satisfied by the layer potentials and their normal derivative.

We recall the standard notations

$$f|_{\pm}(p) = \lim_{t\to 0^+} f(x \pm t\nu_p), \quad p \in \partial\Omega,$$

and

$$\frac{\partial}{\nu_p} f|_{\pm}(p) = \lim_{t\to 0^+} \langle \nabla f(p \pm t\nu_p), \nu_p \rangle, \quad p \in \partial\Omega.$$

We quote from [5, Lemma 11.1, page 186] the following result.

Theorem 5.29. *Let Ω be a C^3 domain in $\mathbb{R}^2$ and let $f \in L^2(\partial\Omega)$. We have*

$$\begin{array}{lcl} (S_k(f))|_+(p) & = & (S_k(f))|_-(p) = S_k(f)(p), \quad a.e \quad p \in \partial\Omega, \\ \left(\dfrac{\partial}{\partial\nu_p} S_k(f)\right)\Big|_{\pm}(p) & = & \left(\pm\frac{1}{2}I + K_k^*\right) f(p), \quad a.e \quad p \in \partial\Omega, \\ D_k(f)|_{\pm}(p) & = & \left(\mp\frac{1}{2}I + K_k\right) f(p), \quad a.e \quad p \in \partial\Omega, \end{array} \tag{5.102}$$

where K_k is the operator defined by

$$K_k\phi(p) := p.v \int_{\partial\Omega} \frac{\partial G_k(p,q)}{\partial\nu_q}\phi(q)d\sigma(q), \quad p \in \partial\Omega, \tag{5.103}$$

and where K_k^* is its $L^2(\partial\Omega)$-adjoint.

An other operator will be of interest and will play a major role in our computations. It is the normal derivative of $D_k(f)$,

$$E_k(f)(p) := \frac{\partial}{\partial\nu_p}\left(\int_{\partial\Omega} \frac{\partial G_k}{\partial\nu_q}(p,q)f(q)\ d\sigma(q)\right), \quad p \in \partial\Omega. \tag{5.104}$$

Remark 5.24. There is not a jump relation for the normal derivative of the double-layer potential across the boundary $\partial\Omega$.

5.8.1.3 Mapping properties in Sobolev spaces

The following results are also needed (see [101, Chapter 7] and [31, Chapter 3]).

Theorem 5.30. *Let Ω be a C^3 domain and $s \in \mathbb{R}$. Then,*

(i) the operator S_k is bounded from $H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$,

(ii) the operators K_k and its adjoint K_k^* are bounded from $H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$,

(iii) the operators $\frac{I}{2} \pm K_k^*$ and $\frac{I}{2} \pm K_k$ are bounded from : $H^s(\partial\Omega)$ into $H^s(\partial\Omega)$,

(iv) the operator $K_k^* - K_0^*$ is continuous from $H^s(\partial\Omega)$ into $H^{s+3}(\partial\Omega)$.

(v) the operator E_k is bounded from $H^s(\partial\Omega)$ into $H^{s-1}(\partial\Omega)$.

Proof of Theorem 5.30. The results concerning the single and double layer potential are developed in (cf. [78, Chapter 4 paragraph 2])) where are studied the boundedness properties of singular integral operators whose kernels are the restriction to $\partial\Omega$ of kernels defined in $\mathbb{R}^2$. In $\mathbb{R}^2$, the layer potential kernel associated to Helmholtz equation is $K(x) = H_0^{(1)}(k \mid x \mid)$ where $H_0^{(1)}$ is the Hankel function of order 0. A Taylor expansion shows that the kernel is

pseudo-homogeneous of classe -1. Thanks to [78, Theorem 4.3.1], we conclude that S_k is bounded from $H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$,
Concerning the double layer potential, its regularity property is due to the fact that its kernel is pseudo-homogeneous of class -1. From [78, Theorem 4.3.1], K_k and K_k^* are bounded from $H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$ for every real s. We point out that one can find the detailed computations in [78, Example 4.5, section 4.3.3].
A Taylor expansion shows that the kernel of the operator $K_k^* - K_0^*$ has the same property as $E(x,y)$, the kernel of the single layer potential corresponding to the biharmonic equation (cf. [36]). We recall that

$$\frac{\partial E(x,y)}{\partial \nu_y} = \frac{1}{8\pi}\langle \nu_y, x-y\rangle \left(2\ln \mid x-y \mid +1\right).$$

The factor $\langle \nu_y, x-y\rangle$ is regular on $\partial\Omega\times\partial\Omega$ and furthermore for small $\mid x-y \mid$ it satisfies

$$\langle \nu_y, x-y\rangle = O(\mid x-y \mid^2). \tag{5.105}$$

Thus, for an element (x,y) living near the diagonal $\partial\Omega\times\partial\Omega$, we have

$$\frac{\partial E(x,y)}{\partial \nu_y} = O(\mid x-y \mid^2 \ln \mid x-y \mid).$$

It follows that $E(x,y)$ and the kernel of $K_k^* - K_0^*$ have the same smoothing effects. Furthermore, from [78, Example 4.3, page 216], we get that the kernel of $K_k^* - K_0^*$ is pseudo-homogeneous of class -3. Thanks to [78, Theorem 4.3.1], it comes that $K_k^* - K_0^*$ is continuous from $H^s(\partial\Omega)$ into $H^{s+3}(\partial\Omega)$, for every real s.
To finish, we see E_k as a pseudodifferential operator on $\partial\Omega$ whose leading symbol is of the form $p(\xi) = -\frac{1}{2}\mid \xi \mid$. Consequently, the operator E_k is continuous from $H^s(\partial\Omega)$ into $H^{s-1}(\partial\Omega)$.
□

5.8.2 Dirichlet-to-Neumann map

The goal of this section is the study of the singularities of the normal derivative of the solution of (5.96). From (5.99), (5.102) and (5.104), we deduce

$$\left(\frac{1}{2}I + K_k^*\right)\frac{\partial u}{\partial \nu} = E_k(f) + \frac{\partial}{\partial \nu}(F * G_k). \tag{5.106}$$

In subsection 5.8.2.1, we study the inverse of the operator $(\frac{1}{2}I + K_k^*)$, thanks to the reduced resolvent theory. In subsection 5.8.2.2, we study the normal derivative of the double-layer potential, $E_k(f)$.

5.8.2.1 Singular perturbation problem and reduced resolvent

Notice that, when $-k^2$ is an eigenvalue of the interior Dirichlet problem for the Laplacian, the integral equation (5.106) is not invertible. The associated operator $(\frac{1}{2}I + K_k^*)$ is in fact invertible except for these critical values.

In this subsection, we show how to solve (5.106) in an efficient manner. More precisely, we consider a general right-hand side v, which is assumed to belong to the range of $\frac{1}{2}I + K_k^*$ and whose Taylor expansion in an open neighborhood of zero takes the following form,

$$v(\sigma) = \alpha_1 \mathrm{p.v}\left(\frac{1}{\sigma}\right) + \alpha_2 \ln|\sigma| + \alpha_3 \sigma \ln|\sigma| + \mathcal{R}_2. \tag{5.107}$$

where α_1, α_2 and α_3 are arbitrary real numbers and where σ denotes the oriented counterclockwise arc-length of the boundary $\partial\Omega$ and $\mathcal{R}_2$ is an error term defined in Definition 5.13.

The main idea is to break up the explicit formula of $\frac{\partial u}{\partial \nu}$ into two parts. The first part reflects the singular behavior of $\frac{\partial u}{\partial \nu}$ and it will not depend on the eigenvalue $-k^2$ of the Laplacian. The second part is a regular remainder of the type $\mathcal{R}_2$. Precisely, the goal of this subsection is the proof of the following result.

Theorem 5.31. *Assume that $\frac{\partial u}{\partial \nu}$ satisfies the equation*

$$\left(\frac{1}{2}I + K_k^*\right)\frac{\partial u}{\partial \nu} = v, \tag{5.108}$$

where v is given by (5.107). Then, we have

$$\frac{\partial u}{\partial \nu} = 2v + T_0 v + \mathcal{R}_2, \tag{5.109}$$

where the linear operator T_0 given by

$$T_0 := -2\left(\frac{1}{2}I + K_0^*\right)^{-1} K_0^*. \tag{5.110}$$

defines a bounded operator from $H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$, for every $s \in \mathbb{R}$.

For the proof of the result, precise information on $(K_k\neg^* - K_0^*)\text{p.v.}(\frac{1}{\sigma})$ is needed. Although we know the higher smoothing effect of $K_k^* - K_0^*$ the operator, we have to show the following result.

Lemma 5.32. *Let $k > 0$. The distribution $(K_k^* - K_0^*)p.v.(\frac{1}{\sigma})$ is of the type $\mathcal{R}_2$.*

Note that the above distribution makes sense thanks to Remark 5.17.

Proof of Lemma 5.32. We are led to study the Taylor expansion of

$$I(\sigma_0) = \int_{-L}^{L} (\sigma - \sigma_0)^2 \ln|\sigma - \sigma_0| \text{p.v.}\left(\frac{1}{\sigma}\right)\, d\sigma, \tag{5.111}$$

for σ_0 in an open neighborhood of zero. We may assume $\sigma_0 > 0$ and we fix $\alpha > 0$ small enough. We have to evaluate the following limit.

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left(\int_{-\alpha}^{-\epsilon}(\sigma-\sigma_0)^2\ln|\sigma-\sigma_0|\frac{1}{\sigma}\,d\sigma + \int_{\epsilon}^{\alpha}(\sigma-\sigma_0)^2\ln|\sigma-\sigma_0|\frac{1}{\sigma}\,d\sigma\right)\\
=\;& \lim_{\epsilon\to 0}\int_{\epsilon}^{\alpha}\frac{d\sigma}{\sigma}[(\sigma-\sigma_0)^2\ln|\sigma-\sigma_0| - (\sigma+\sigma_0)^2\ln|\sigma+\sigma_0|]\\
=\;& I_1(\sigma_0) + I_2(\sigma_0) + I_3(\sigma_0),
\end{aligned}$$

where we set

$$\begin{aligned}
I_1(\sigma_0) &:= \lim_{\epsilon\to 0}\int_{\epsilon}^{\alpha}\sigma\ln\frac{|\sigma-\sigma_0|}{\sigma_0+\sigma}\,d\sigma,\\
I_2(\sigma_0) &:= -2\sigma_0\lim_{\epsilon\to 0}\int_{\epsilon}^{\alpha}(\ln|\sigma-\sigma_0| + \ln|\sigma+\sigma_0|)d\sigma\\
I_3(\sigma_0) &:= \sigma_0^2\lim_{\epsilon\to 0}\int_{\epsilon}^{\alpha}\ln\frac{|\sigma-\sigma_0|}{\sigma_0+\sigma}\frac{d\sigma}{\sigma}.
\end{aligned}$$

We first estimate $I_1(\sigma_0)$. The function in the integral is integrable at $\sigma = 0$, then $I_1(\sigma_0) = \displaystyle\int_0^\alpha \sigma \ln \frac{|\sigma - \sigma_0|}{\sigma_0 + \sigma}\, d\sigma$. We first make the change of variable $t = \sigma/\sigma_0$. We get $I_1(\sigma_0) = \sigma_0^2(C_0 + J_1(\alpha/\sigma_0))$, where $C_0 = \displaystyle\int_0^1 t \ln \frac{|t-1|}{t+1} dt$ and $J_1(X) = \displaystyle\int_1^X t \ln \frac{|t-1|}{t+1} dt$ for $X \geq 1$. By integrating by part J_1, we obtain

$$I_1(\sigma_0) = C_0\sigma_0^2 + \frac{\alpha^2 - \sigma_0^2}{2} \ln\Big(\frac{\alpha - \sigma_0}{\alpha + \sigma_0}\Big) - \sigma_0(\alpha - \sigma_0).$$

Then I_1 is of class C^2 in a neighborhood of zero.

We next consider $I_2(\sigma_0)$. We have

$$\begin{aligned} I_2(\sigma_0) &= -2\sigma_0 \left\{ \int_{-\sigma_0}^{\alpha-\sigma_0} \ln|s| ds + \int_{\sigma_0}^{\alpha+\sigma_0} \ln|s| ds \right\} \\ &= -2\sigma_0 \left\{ (\alpha - \sigma_0) \ln|\alpha - \sigma_0| + (\alpha + \sigma_0) \ln|\alpha + \sigma_0| - 2\alpha \right\}, \end{aligned}$$

which show that I_2 is real analytic in an open neighborhood of zero.

Finally, we estimate $I_3(\sigma_0)$. We have

$$\begin{aligned} I_3(\sigma_0) &= \sigma_0^2 \lim_{\epsilon\to 0} \int_\epsilon^\alpha \ln \frac{|\sigma - \sigma_0|}{\sigma_0 + \sigma} \frac{d\sigma}{\sigma} = \sigma_0^2 \lim_{\epsilon\to 0} \int_{\epsilon/\sigma_0}^{\alpha/\sigma_0} \ln \frac{|1-t|}{1+t} \frac{dt}{t} \\ &= \sigma_0^2 C_1 + \sigma_0^2 H_1(\alpha/\sigma_0), \end{aligned}$$

where $C_1 = \int_0^1 \ln \frac{1-t}{1+t} \frac{dt}{t}$ and $H_1(X) = \int_1^X \ln \frac{t-1}{t+1} \frac{dt}{t}$ for $X \geq 1$. Making the change of variable $v = \frac{t-1}{t+1}$ in H_1, we have $H_1(\alpha/\sigma_0) = 2\int_0^\beta \frac{\ln v}{1-v^2} dv$, where $\beta = \frac{\alpha-\sigma_0}{\alpha+\sigma_0}$. We note that $\beta < 1$. Then,

$$\begin{aligned} \int_0^\beta \frac{\ln v}{1-v^2} dv &= \int_0^\beta \ln v \sum_{n\geq 0} v^{2n} dv = \sum_{n\geq 0} \int_0^\beta (\ln v) v^{2n} dv \\ &= \sum_{n\geq 0} \frac{\beta^{2n+1} \ln\beta}{2n+1} - \sum_{n\geq 0} \frac{\beta^{2n+1}}{(2n+1)^2} = S_1 + S_2 \end{aligned}$$

For S_1, we have

$$\begin{aligned} S_1 &= \frac{1}{2}\ln(\beta) \left(\sum_{n\geq 1} \frac{\beta^n}{n} - \sum_{n\geq 1} \frac{(-\beta)^n}{n} \right) \\ &= \frac{1}{2}\ln(\beta) \Big(-\ln(1-\beta) + \ln(1+\beta) \Big) \\ &= \frac{1}{2} \ln\frac{1+\beta}{1-\beta} \ln\beta \\ &= \frac{1}{2} \ln\left(1 - \frac{2\sigma_0}{\alpha + \sigma_0}\right) \ln\frac{\alpha}{\sigma_0}. \end{aligned}$$

For S_2, we begin by computing $\frac{dS_2}{d\sigma_0}$.

$$\begin{aligned} \frac{dS_2}{d\sigma_0} &= -\sum_{n\geq 0} \frac{\beta^{2n}}{2n+1} \frac{d\beta}{d\sigma_0} = \frac{1}{\beta} \sum_{n\geq 0} \frac{\beta^{2n+1}}{2n+1} \frac{2\alpha}{(\alpha+\sigma_0)^2} \\ &= \frac{\alpha}{\alpha^2 - \sigma_0^2} \ln\frac{\alpha}{\sigma_0} = \frac{1}{\alpha(1 - \frac{\sigma_0^2}{\alpha^2})} \ln\frac{\alpha}{\sigma_0}. \end{aligned}$$

Recall that $I_3(\sigma_0) = \sigma_0^2 C_1 + \sigma_0^2(S_1 + S_2)$, the computations above show that I_3 is a 2-regular term.

□

We are now ready to prove Theorem 5.31

Proof of Theorem 5.31. We subdivide the proof in several steps.
Step 1 : We begin to recall some results on the reduced resolvent theory (cf. [57, Chapter I paragraph 5]). Since $\lambda = 0$ is an eigenvalue of $\left(\frac{1}{2}I + K_k^*\right)$, the resolvent

$$R(\lambda) = \left(\left(\frac{1}{2} - \lambda\right) I + K_k^*\right)^{-1}$$

has a singularity at $\lambda = 0$. Since the dimension of the eigenspace associated to $\lambda = 0$ is equal to one, the resolvent is expanded as a Laurent series

$$\left(\left(\frac{1}{2} - \lambda\right) I + K_k^*\right)^{-1} = \frac{A_{-1,k}}{\lambda} + \sum_{n=0}^{\infty} \lambda^n A_{0,k}^{n+1}$$

in a neighbourhood of $\lambda = 0$. The notations $A_{-1,k}$ and $A_{0,k}$ stand for

$$A_{-1,k} := \frac{1}{2i\pi} \int_\Gamma \left(\left(\frac{1}{2} - \lambda\right) I + K_k^*\right)^{-1} \, d\lambda,$$

and

$$A_{0,k} := \frac{1}{2i\pi} \int_\Gamma \frac{1}{\lambda} \left(\left(\frac{1}{2} - \lambda\right) I + K_k^*\right)^{-1} \, d\lambda,$$

where Γ is a small positively oriented circle enclosing 0 in $\mathbb{C}$. According to [57], the operator $P_0 := -A_{-1,k}$ is a projector on the null space associated to $\lambda = 0$ and moreover

$$\begin{array}{ccccc} A_{0,k}P_0 & = & P_0 A_{0,k} & = & 0, \\ \left(\frac{1}{2}I + K_k^*\right) A_{0,k} & = & A_{0,k}\left(\frac{1}{2}I + K_k^*\right) & = & I - P_0. \end{array}$$

The last equalities show that $A_{0,k}$ is the "inverse" of $\left(\frac{1}{2}I + K_k^*\right)$ restrained to the complementary subspace to the null space associated to $\lambda = 0$.
Step 2 : Using the reduced resolvent method, one gets

$$\frac{\partial u}{\partial \nu} = A_{0,k} v + \mathcal{U}(v), \tag{5.112}$$

where $\mathcal{U}(\sqsubseteq)$ is an arbitrary element belonging to Ker $\left(\frac{1}{2}I + K_k^*\right)$.

Recall that Ker $\left(\frac{1}{2}I + K_k^*\right)$ coincides with the span of the traces of all normal derivatives on $\partial\Omega$ of Dirichlet eigenfunctions of the Laplacian with eigenvalue $-k^2$ (see [34, page 684]). We then deduce that $\mathcal{U}(\sqsubseteq)$ is of type $\mathcal{R}_2$.

We can now rewrite Equation (5.112) as follows

$$\begin{aligned}
\frac{\partial u}{\partial \nu} &= \left(\tfrac{1}{2}I + K_0^*\right)^{-1} v + \left[A_{0,k} - \left(\frac{1}{2}I + K_0^*\right)^{-1}\right] v + \mathcal{U}(v) \\
&= \left(\tfrac{1}{2}I + K_0^*\right)^{-1} v \\
&\quad + \frac{1}{2i\pi}\int_\Gamma \frac{d\lambda}{\lambda}\left[\left(\left(\frac{1}{2} - \lambda\right) I + K_k^*\right)^{-1} - \left(\frac{1}{2}I + K_0^*\right)^{-1}\right] v \\
&\quad + \mathcal{U}(v) \\
&= \left(\tfrac{1}{2}I + K_0^*\right)^{-1} v \\
&\quad + \frac{1}{2i\pi}\int_\Gamma \frac{d\lambda}{\lambda}\left[\left(\left(\frac{1}{2} - \lambda\right) I + K_k^*\right)^{-1} (K_0^* - K_k^* + \lambda I)\left(\frac{1}{2}I + K_0^*\right)^{-1}\right] v \\
&\quad + \mathcal{U}(v) \\
&= \left(\tfrac{1}{2}I + K_0^*\right)^{-1} v + A_{0,k}\left[(K_0^* - K_k^*)\left(\frac{1}{2}I + K_0^*\right)^{-1}\right] v \\
&\quad + A_{-1,k}\left[\left(\tfrac{1}{2}I + K_0^*\right)^{-1} v\right] + \mathcal{U}(v).
\end{aligned} \tag{5.113}$$

Writing

$$\left(\frac{1}{2}I + K_0^*\right)^{-1} = 2I - 2\left(\frac{1}{2}I + K_0^*\right)^{-1} K_0^*,$$

it follows that

$$\frac{\partial u}{\partial \nu} = 2v - 2\left(\frac{1}{2}I + K_0^*\right)^{-1} K_0^* v + \mathcal{V}(v) + \mathcal{W}(v),$$

where

$$\mathcal{V}(v) := A_{0,k}\left[(K_0^* - K_k^*)\left(\frac{1}{2}I + K_0^*\right)^{-1}\right] v,$$

and

$$\mathcal{W}(v) := A_{-1,k}\left[\left(\frac{1}{2}I + K_0^*\right)^{-1}\right] v + \mathcal{U}(v).$$

Since $-A_{-1,k}$ is a projector on the null eigenspace associated to the zero eigenvalue, the remainder $\mathcal{W}(v)$ belongs to $\mathcal{R}_2$. The smoothing effects of $K_k^* - K_0^*$ described in Lemma 5.32 and Theorem 5.30(iv) show that $\mathcal{V}(v)$ belongs also to $\mathcal{R}_2$. Concerning the term $T_0 v$, its regularity is deduced from the fact that

$$\frac{1}{2}I + K_0^* \ : \ H^s(\partial\Omega) \to H^s(\partial\Omega)$$

is an isomorphism and that K_0^* is a bounded operator from $H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$ for every real s.

□

5.8.2.2 Normal derivative of the double-layer potential

In [34], the normal derivative of a double-layer potential is investigated in dimension three. For our purpose, we adapt their computations in dimension two.

Lemma 5.33. *Let $k \in \mathbb{C}$ with $Im\ k \geq 0$ and $f \in \mathcal{D}'(\partial\Omega)$. We have, for every ψ in $\mathcal{D}(\partial\Omega)$,*

$$\langle E_k(f), \psi\rangle = -\langle G_k * \frac{\partial f}{\partial \tau}, \frac{\partial \psi}{\partial \tau}\rangle + k^2 \int_{\partial\Omega} \psi(p) \int_{\partial\Omega} f(q) G_k(p,q) \langle \nu_q, \nu_p\rangle d\sigma(q) d\sigma(p),$$

where $\langle ., .\rangle$ refers to the $\mathcal{D}'(\partial\Omega)/\mathcal{D}(\partial\Omega)$-duality, and $*$ is the convolution product on $\partial\Omega$.

Remark 5.25. For details about the convolution product defined on $\partial\Omega$, we can refer to [88, Chapitre IV, page 166-168].

Lemma 5.34. *Let $f := \mathcal{H}_0 g$ where $\mathcal{H}_0$ is the Heaviside function with a jump at zero and $g : \partial\Omega \to \mathbb{R}$ is smooth. We have*

$$E_k(f)(p(\sigma)) = g(0)\frac{\partial G_k}{\partial \tau}(p(\sigma)) + \frac{\partial g}{\partial \tau}(0) G_k + G_k * \mathcal{H}_0 \left(\frac{\partial^2 g}{\partial \tau^2} + k^2 g\right) + O(\sigma^2), \tag{5.114}$$

in the space of distributions $\mathcal{D}'(\partial\Omega)$.

Proof of Lemma 5.34. We apply Lemma 5.33 to $f = \mathcal{H}_0 g$. On the one hand, we have

$$\begin{aligned} -\langle G_k * \frac{\partial f}{\partial \tau}, \frac{\partial \psi}{\partial \tau}\rangle &= -\langle G_k * \left(\delta_0 g(0) + \mathcal{H}_0 \frac{\partial g}{\partial \tau}\right), \frac{\partial \psi}{\partial \tau}\rangle \\ &= g(0)\langle \frac{\partial G_k}{\partial \tau}, \psi\rangle + \frac{\partial g}{\partial \tau}(0)\langle G_k, \psi\rangle + \langle G_k * \left(\mathcal{H}_0 \frac{\partial^2 g}{\partial \tau^2}\right), \psi\rangle. \end{aligned}$$

On the other hand, using $\langle \nu_{p(0)}, \nu_{q(\sigma)}\rangle = 1 + O(\sigma^3)$ in a neighborhood of $\sigma = 0$, we get

$$\begin{aligned} & k^2 \int_{\partial\Omega} \psi(p) \int_{\partial\Omega} f(q) G_k(p,q) \langle \nu_q, \nu_p\rangle d\sigma(q) d\sigma(p) \\ = \ & k^2 \int_{\partial\Omega} \psi(p) \int_{\partial\Omega} f(q) G_k(p,q) [1 + O(\sigma^2)] d\sigma(q) d\sigma(p) \\ = \ & \langle k^2 G_k * f, \psi\rangle + \langle O(\sigma^2), \psi\rangle. \end{aligned}$$

□

Then, we have the following result.

Proposition 5.35. *Let $f = \mathcal{H}_0 g$, with $\mathcal{H}_0$ the Heaviside function with jump at zero and $g : \partial\Omega \to \mathbb{R}$ a smooth function. We have*

$$\begin{aligned} E_k(f)(p(\sigma)) &= g(0) a_1 p.v.\left(\frac{1}{\sigma}\right) + \frac{\partial g}{\partial \tau}(0) a_1 \ln|\sigma| \\ &\quad + \left\{a_1 \frac{\partial^2 g}{\partial \tau^2} + (a_1 + 2a_2) k^2 g\right\}(0) \sigma \ln|\sigma| + \mathcal{R}_2, \end{aligned} \tag{5.115}$$

where a_1 and a_2 are defined in (5.62)

Proof of Proposition 5.35. According to Proposition 5.28, we get

$$G_k(p,0) = a_1 \ln|p| + c_k + a_2 k^2 |p|^2 \ln|p| + O(|p|^2) \text{ when } p \to 0 \tag{5.116}$$

where c_k is a constant depending on k^2. Thus, using (5.57) and (5.58) we get

$$G_k(p(\sigma),0) = a_1 \ln|\sigma| + c_k + a_2 k^2 \sigma^2 \ln|\sigma| + O(\sigma^2). \tag{5.117}$$

Similar computations show that

$$\frac{\partial G_k}{\partial \tau}(p(\sigma),0) = a_1 \text{p.v.}\left(\frac{1}{\sigma}\right) + 2a_2 k^2 \sigma \ln|\sigma| + O(\sigma). \tag{5.118}$$

Consider $\tilde{g} = \frac{\partial^2 g}{\partial \tau^2} + k^2 g$ and calculate now $G_k * \mathcal{H}_0 \tilde{g}$. Since G_k is a compactly supported distribution, then $G_k * \mathcal{H}_0$ is a primitive of G_k (see for example [88, Chapitre IV, page 168]). Thanks to the fact that $\tilde{g}$ is a smooth function, we conclude that

$$G_k * \mathcal{H}_0 \tilde{g} = \tilde{g}(0) a_1 \sigma \ln|\sigma| + \alpha_k + O(\sigma), \tag{5.119}$$

where α_k is a constant of integration. This ends the proof of the proposition. □

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Résumé : L'objectif de cette thèse est, d'une part, de fournir des méthodes de planification de mouvements pour les systèmes non-holonomes, et d'autre part, d'étudier la contrôlabilité spectrale pour les équations de Schrödinger linéarisées.
Nous avons apporté une double contribution au problème de la planification de mouvements pour les systèmes non-holonomes. Fondé sur la géométrie sous-riemannienne, nous avons conçu un nouvel algorithme qui résout complètement le problème dans un cadre général. Nous avons également proposé une implémentation numérique de la méthode de continuation qui fournit des solutions satisfaisantes au problème de la planification du roulement sur le plan, un exemple classique de systèmes non-holonomes à deux entrées.
Nous avons donné des conditions nécessaires et suffisantes de contrôlabilité spectrale en temps fini des équations de Schrödinger linéarisées en dimension 2 et 3. Leur généricité par rapport au domaine a été étudiée par une technique originale basée sur les équations intégrales.

Mots clés : Planification de mouvements, systèmes non-holonomes, géométrie sous-riemannienne, approximation nilpotente, méthode de continuation, problème de roulement, équation de Schrödinger, contrôlabilité spectrale, minimalité des familles exponentielles, contrôlabilité générique, différentiation de forme, équation de Helmholtz, représentation intégrale

Abstract : The objective of this thesis is, firstly, to provide motion planning algorithms for nonholonomic systems, and secondly, to study the spectral controllability for the linearized Schrödinger equations.
We made a double contribution to the problem of motion planning for nonholonomic systems. Based on the sub- Riemannian geometry , we have developed a new algorithm that completely solves the problem in a general framework. We have also proposed a numerical implementation of the continuation method that provides satisfactory solutions to the rolling-body problem, a classic example of nonholonomic systems with two inputs.
We have given necessary and sufficient conditions of spectral controllability in finite time for the linearized Schrödinger equations in dimension 2 and 3. Their genericity with respect to the domain has been studied by a novel technique based on integral equations.

Key words : Motion planning, nonholonomic systems, sub-Riemannian geometry, nilpotent approximation, continuation method, rolling-body problem, Schrödinger equation, spectral controllability, minimality of exponential families, generic controllability, shape differentiation, Helmholtz equation, integral representation